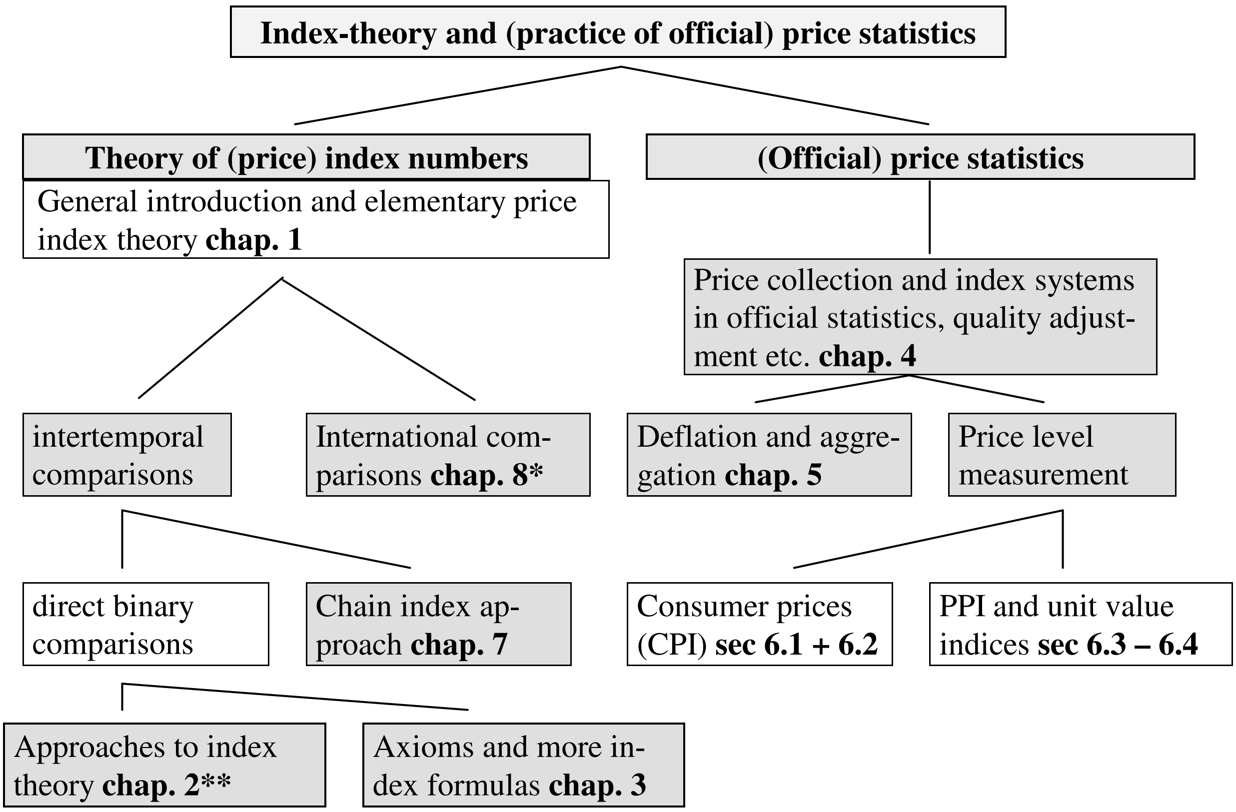
Prof. Dr. Peter von der Lippe

Structure of the MEDSTAT Course and of the book "Index Theory and Price Statistics"

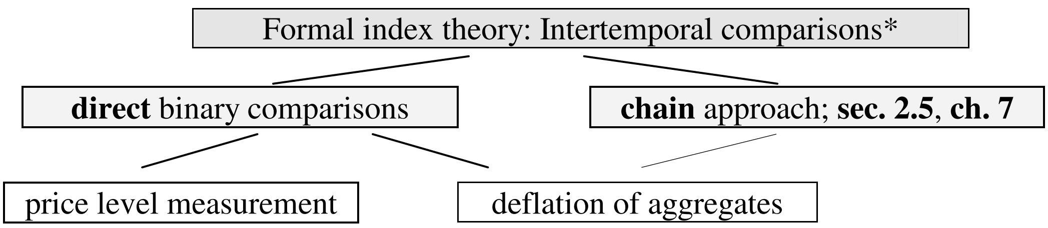
Figure 1.1.1 a



* not presented in this course

** e.g. formal index theory (focusing on mathematical properties of index functions) and economic theory of index numbers (aiming at a microeconomic foundation of index formulas).

Figure 1.1.1 b: Structure of formal index theory



Next page: structure of the book (without section 6.5: Employment cost index)

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Chapter 1 General introduction and elementary price index theory

1.1. Fundamental principles

a) The concept of a price index	c) Simple comparisons (a single commodity)
b) Objectives and methodological principles of price statistics, the concept of deflation	d) Aggregative comparisons (two or more commodities) and unit values

a) The concept of a price index

Definition: A price index P_{0t} is a function P: $\mathbb{R}^{kn} \rightarrow \mathbb{R}$ mapping k = 4 real valued vectors with n dimensions

(1.1.1) $P_{0t} = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$ (does **not** apply to the "economic theory" index [= True Cost-of-Living Index] nor to chain indices)

into a one dimensional positive real number for comparative purposes. The function P( ) should satisfy certain functional equations (= axioms) and (in order to) have a meaningful interpretation.

The word **base** is ambiguous because it may refer to the period to which

1. we compare the current state (**reference base**), or to which
2. the weights refer (**weight base**).

b) Objectives and methodological principles of price statistics, the concept of deflation

Deflation: Aggregate at **current** (period t) prices,

(1.1.2.) $V_t = \sum_i p_{it} q_{it}$

which is called a **value,** V_t (or "nominal" aggregate). The same (with respect to the selection of commodities and their quantities) aggregate valued at **constant** prices of the base period 0

(1.1.3) $Q_t = \sum_i p_{i0} q_{it}$ is called a **volume** (or "real" aggregate)

(1.1.4) $V_{0t} = V_t / V_0$ **value index** (value ratio).

Fundamental methodological principles of price statistics and price indexes (see fig. 1.1.2)

a) the selection of "representative" prices, and

b) the principle of "pure price comparison".

To better understand the two conflicting principles **representativity (R)** and **pure price comparison (P)** we should start defining "**Price determining characteristics**" (**PDC**). The PDCs are the quantity and quality of the commodity, the shop (outlet) in which the sale takes place, a bonus granted or services rendered in connection with the sale if applicable, arrangements made concerning delivery, availability of spare parts, insurance etc.

As conditions inevitably change, obviously both principles are hard to reconcile In practice always some *compromise* in one way or another between R and P has to be found which tries to comply with *both* conflicting principles acceptably. No solution can meet each of them to full satisfaction.

Note that this conflict also applies to index formulas: A reasonable compromise is - in our view - neither a chain index (focusing on R at the expense of P) nor keeping weights of a Laspeyres index unchanged for quite a long period in time (that is pursuing exclusively P and neglecting R unduly) but rather a Laspeyres index in which weights are reviewed and updated in intervals of say five years or so.

Figure 1.1.2: Uses of price statistics and price indices

a) Purposes of price indices in intertemporal comparisons

analytical use: the measurement of "price levels" on certain markets

deflation: the estimation of the underlying quantity of an aggregate, or of real income

b) Methodological principles of price indices and price statistics
All price determining characteristics (**PDC**) of contracts have to be taken into account

representativeness* (R)	**pure price comparison (P)**
select contracts governed by the most common (frequent) and typical PDCs	PDCs prevailing in both situations to be compared should be as similar as possible

* or: representativity

c) Simple comparisons (a single commodity)

Three ways of describing the change in an individual price of commodity i:

1. ***price relatives*** (≈ growth *factors*) with reference to a base period price p_{i0} (**eq. 1.1.5**) and links (eq. 1.1.6),
2. ***log-changes*** (≈ growth *rates* referring to a logarithmic mean as the reference value) → **eq. 1.1.8**, → leading to "log-change indices" (sec. 3.4 and 3.5)
3. ***differentials*** with respect to time $dp_i(t)/dt$ terms → Divisia index

3a. and (less common) **absolute differences** (= variations, Hillinger).

Relatives (related to a fixed base) as opposed to links (link relatives) = variable base

(1.1.5) $$a_{0t} = a^i_{0t} = p_{it}/p_{i0}\ .$$

Quantity relative $b_{0t} = \frac{q_t}{q_0}$ and a value relative $c_{0t} = v_{0t} = \frac{v_t}{v_0} = \frac{p_t q_t}{p_0 q_0} = a_{0t} b_{0t}$.

Links (chain base) and Log changes

(1.1.6) $$\ell_t = p_t/p_{t-1} = a_{t-1,t}$$

(1.1.7) $$a_{0t} = \ell_1 \ell_2 \ldots \ell_{t-1} \ell_t\ .$$

The terms

(1.1.8) $Da_{0t} = \ln(p_t/p_0)$ or (1.1.8a) $D\ell_t = \ln(p_t/p_{t-1})$.

are called **log-changes**. There exists an appropriate mean of terms like Da_{0t}, that is the logarithmic mean of two positive numbers, x and y (x ≠ y)

(1.1.9) $L(x,y) = \frac{y-x}{\ln(y/x)} = L(y,x)$ logarithmic mean.

(1.1.9a) $\ln\left(\frac{p_{it}}{p_{i0}}\right) = \frac{p_{it}-p_{i0}}{L(p_{pit},p_{pit})}$ Log changes can be interpreted as growth rates.

There are some index formulas based on log changes and L, like the Törnquist index (P^T) or Vartia indices, but (with the exception of P^T their role in official statistics was rather small until now.

Table 1.1.1: Axioms satisfied by price and quantity relatives (fixed base)

no	axiom	definition and interpretation	
1	**identity**	$a_{00} = a_{tt} = 1$	uniqueness of the reference point
2	**dimensionality** (price dimensionality)	$\frac{\lambda p_t}{\lambda p_0} = \frac{p_t}{p_0}$	a_{0t} is independent of the currency in which the prices are expressed [b)]
3	**commensurability** [a)]	$\frac{\lambda p_t}{\lambda p_0} = \frac{p_t}{p_0}$	independence of the unit of quantity to which the price of commodity i refers [b)]
4	**time reversal test**	$a_{t0} = \frac{1}{a_{0t}}$	consistency of relatives with different base periods
5	**factor reversal test**	$c_{0t} = a_{0t}b_{0t}$	the value change is decomposable in price change and quantity change
6	**transitivity**	$a_{0t} = a_{0s}a_{st}$	for *all* three periods 0, s and t

a) if only n = 1 commodity is involved the mathematical representation of 2 and 3 can not be distinguished.
b) in both periods, 0 and t.

d) Aggregative comparisons (two or more commodities) and unit values

(1.1.10) $\tilde{p}_t = \frac{\sum p_{it}q_{it}}{\sum q_{it}} = \sum p_{it}\frac{q_{it}}{\sum q_{it}}$ (unit value in t)

Example 1.1.1

Imagine an economy with only two industries A and B, and wages of \$10 and \$16 paid at base period:

situation in base period			
industry	wage	hours	payment
A	10	50	500
B	16	50	800
sum*	13	100	1300

* or average

It would be highly misleading to compare simply the average wage per hour presently paid with the average wage formerly paid at the base year. Assume two alternative (presented for demonstrative purposes) situations in t

situation in t			
industry	wage	hours	payment
A	15	90	1350
B	24	10	240
total	15.9	100	1590

alternative situation in t			
industry	wage	hours	payment
A	15	10	150
B	24	90	2160
total	23.1	100	2310

Structural change has to be eliminated by some method of weighting (for example with a constant base year structure, like in a Laspeyres type index). Indices differ from averages ("unit values") by their invariance to a structural change and thus their ability to provide "pure" comparisons.

1.2. Unweighted indices

a) Dutot's index and Drobisch's unit value index	c) Commensurability and time reversal test
b) Carli's index formula	d) Stochastic and aggregative approach

a) Dutot's index and unit value index of Drobisch

(1.2.1) $$P_{0t}^{D} = \frac{\bar{p}_t}{\bar{p}_0} = \frac{\frac{1}{n}\sum p_{it}}{\frac{1}{n}\sum p_{i0}} = \frac{\sum p_{it}}{\sum p_{i0}}$$ (price index of **Dutot** 1738).

(1.2.2) $$P_{0t}^{UD} = \frac{\tilde{p}_t}{\tilde{p}_0} = \frac{\Sigma p_{it} q_{it} / \Sigma q_{it}}{\Sigma p_{i0} q_{i0} / \Sigma q_{i0}}$$ (unit value index of **Drobisch** 1871).

Both formulas have in common that they are *ratios of absolute figures*, that is of average prices expressed in €, \$, £ or the like. Such indices violate *commensurability*

(reason: The sums $\Sigma p_t q_t$ and $\Sigma p_0 q_0$ are not affected by a change in the physical units of quantities, but the sums Σq_t and Σq_0 are and so are the sums Σp_t and Σp_0.).

P^{UD} has three additional shortcomings (→ ex. 1.2.1):

- P^{UD} does not meet **the mean value condition**,
- the sums $\sum_i q_{it}$ and $\sum_i q_{i0}$ are in general not defined,
- P^{UD} can indicate a change of the price level although all prices remain unchanged, simply because quantity changed (in level or in structure). Hence P^{UD} violates **identity**.

Example 1.2.1

Consider the following prices and quantities of two commodities, A and B

i	p_{i0}	p_{it}	q_{i0}	q_{it}	q_{it}^{*}
A	10	15	5	8	10
B	30	35	2	4	2
Σ	40	50	7	12	12

Index of Dutot: $P_{0t}^{D} = \frac{\bar{p}_t}{\bar{p}_0} = \frac{25}{20} = \frac{\sum p_t}{\sum p_0} = \frac{50}{40} = 1.25$. Now assume prices p_{A0} and p_{At} refer to quarts and price statistics changes to a quotation in terms of gallons (prices and quantities of B remain unchanged). With prices of A on the basis of gallons we get

$P_{0t}^{D} = \frac{60+35}{40+30} = \frac{95}{70} = 1.357$ indicating a rise of prices by more than 25%. The unit value index P^{UD} amounts to $P_{0t}^{UD} = \frac{(120+140)/12}{(50+60)/7} = \frac{260/12}{110/7} = \frac{21.67}{15.71} = 1.3788$.

Assume now prices remained unchanged, so that $p_{At} = 10$ and $p_{Bt} = 30$. A reasonable index should be unity, however $P_{0t}^{UD} = \frac{200/12}{110/7} = \frac{16.67}{15.71} = 1.0606$, indicating a rise by 6%. Since

(1.2.2a) $$P_{0t}^{UD} = \sum p_{it} \frac{q_{it}}{\sum q_{it}} \Big/ \sum p_{i0} \frac{q_{i0}}{\sum q_{i0}} = \frac{\sum p_{it}\alpha_{it}}{\sum p_{i0}\alpha_{i0}}$$

where weights α are reflecting the structure of quantities. As weights α_{it} differ from α_{i0} such that commodity B, which is cheaper than A, gets a weight $\alpha_{it} > \alpha_{i0}$ the result is $P^{UD} < 1$. Taking quantities q* we obtain: $P_{0t}^{UD} = \frac{(100+60)/12}{110/7} = \frac{13.33}{15.71} = 0.8485$. ♦

Note: $P_{0t}^{L} = \frac{\sum p_{it}\alpha_{i0}}{\sum p_{i0}\alpha_{i0}}$; $P_{0t}^{P} = \frac{\sum p_{it}\alpha_{it}}{\sum p_{i0}\alpha_{it}}$

An analysis of P^{UD} also shows why the ratio of "average wages" (virtually unit values with quantities q being numbers of employees) in **ex. 1.1.1** is unacceptable. Since in this example $\Sigma q_t = \Sigma q_0 = 100$ P^{UD} equals the simple value ratio (index) that can be decomposed as follows:

(1.2.3) $$V_{0t} = \frac{\Sigma p_t q_t}{\Sigma p_0 q_0} = \frac{\Sigma p_t q_0}{\Sigma p_0 q_0} + \frac{\Sigma p_t (q_t - q_0)}{\Sigma p_0 q_0} = P_{0t} + S_{0t}.$$

b) Carli's index formula

In order to satisfy the commensurability axiom and to eliminate the structural component a price index should be a mean of ratios (relatives) rather than of a ratio of means.

(1.2.4) $$P_{0t}^{C} = \frac{1}{n}\sum_i \frac{p_{it}}{p_{i0}} = \frac{1}{n}\sum a_{0t}^{i}$$ (price index of **Carli** 1764).

(1.2.5) $$a_{0t}^{min} \le P_{0t}^{C} \le a_{0t}^{max}.$$

c) Commensurability and time reversal test, choice of the type of average in aggregating price quotations

(1.2.6) $$P_{t0} = \frac{1}{P_{0t}}$$, time reversal test

making a price index invariant to a change of the base period. Unfortunately many of the best index theoreticians are setting great store by this property

The emphasis placed on time reversibility as well as the idea that the formulas of Laspeyres and Paasche are "equally justified" (prompting the taking of an average of both formulas) rests on the tacit assumption that periods, 0 and t are having the same logical status. However, 0 and t are *not* equivalent and on the same footing. This can easily be seen as 0 is kept constant for some periods at least, while t denotes a *sequence* of periods, 1, 2, ...

In order to justify the rejection of Carli's formula Haschka 1999 gave the following example:

i	p_{i0}	p_{it}	price relatives
1	20	10	0.5
2	10	20	2
	$\bar{p}_0 = 15$	$\bar{p}_t = 15$	

price index
Dutot $P^D = 15/15 = 1$
Jevons $P^J = \sqrt{0.5 \cdot 2} = 1$
Carli $P^C = (2+0.5)/2 = 1.25$

However

i	p_{i0}	p_{it}	price relatives
1	20	10	0.5
2	100	200	2
	$\bar{p}_0 = 60$	$\bar{p}_t = 105$	

price index
Dutot $P^D = 105/60 = 1.75$
Jevons $P^J = \sqrt{0.5 \cdot 2} = 1$
Carli $P^C = (2+0.5)/2 = 1.25$

i	p_{i0}	p_{it}	price relatives
1	200	210	1.05
2	20	10	0.5
	$\bar{p}_0 = 110$	$\bar{p}_t = 110$	

price index
Dutot $P^D = 1$
Jevons $P^J = 0.72457$
Carli $P^C = 0.775$

Figure 1.2.2: Behavior of unweighted indices by type of mean

unweighted mean of relatives

arithmetic (PA)	geometric (PG)	harmonic (PH)
(1.2.7) $PA_{t0} > (PA_{0t})^{-1}$	(1.2.8) $PG_{t0} = (PG_{0t})^{-1}$	(1.2.9) $PH_{t0} < (PH_{0t})^{-1}$
Carli's index (1.2.4) $P_{0t}^C = \sum (p_{it}/p_{i0}) / n$ $P_{t0}^C > 1/P_{0t}^C$	Jevons' index (1.2.10) $P_{0t}^{JV} = \sqrt[n]{\prod \frac{p_{it}}{p_{i0}}}$	unnamed index (1.2.11) $P_{0t}^{HM} = \frac{n}{\sum \frac{p_{i0}}{p_{it}}}$

These relations also hold for weighted means as well since they refer to the *kind of mean*. Thus for to the Laspeyres index (a PA-type) eq. 7 applies whereas for the Paasche index (PH-type) eq. 9 applies. For PHM see also fig. 2.2.1

(1.2.12) $$PA_{0t} = P_{0t}^C = 1/P_{t0}^{HM}.$$

(1.2.13) $$P_{0t}^D = \sum \left(\frac{p_{i0}}{\sum p_{i0}} \right) \frac{p_{it}}{p_{i0}} \neq P_{0t}^C = \sum \left(\frac{1}{n} \right) \frac{p_{it}}{p_{i0}}.$$

Some new unweighted index functions

The index

(1.2.12a) $P_{0t}^{CSWD} = \sqrt{P_{0t}^C P_{0t}^{HM}}$ is known as CSWD (Carruthers – Sellwood – Ward - Dalen) index. And the index

(1.2.12b) $$P_{0t}^{HYB} = \sum \frac{p_{it}}{p_{i0}} \frac{\sqrt{p_{i0}/p_{it}}}{\sum \sqrt{p_{i0}/p_{it}}} = \frac{\sum p_{it} \sqrt{(1/p_{i0})(1/p_{it})}}{\sum p_{i0} \sqrt{(1/p_{i0})(1/p_{it})}}$$

has been introduced as "hybrid-index" by Jens Mehrhoff in a short note contributed to my book on Index Theory (p. 45 f.). He found the formula as a linear approximation of Jevons index. P^{HYB} is also known as Balk-Walsh index because it corresponds as an unweighted index to the weighted Walsh-Index (eq. 2.2.9) and has been introduced by Bert Balk (2005).

Both indices, CSWD and HYB are special cases of a generalized average (see. eq. 2.2.19). In sec. 2.2.b we will also mention some other index formulas, such as the exponential index.

Table 1.2.1: Some properties of unweighted means as indices [a)]

	commensurability (C)	
time reversal test (T)	satisfied	violated
satisfied	Jevons P^{JV}, CSWD and "Hybrid"-index	Dutot P^D
violated	Carli[b)] P^C	Drobisch P^{UD}

a) including Drobisch's unit value index; all index functions listed violate the factor reversal test
b) the same is true for an unweighted *harmonic* mean.

Given that nowadays (an unwarranted) great store is set by the time reversal condition it is not surprising that in international standards for an unweighted aggregation of price quotations* the formulas P^{JV} and P^D are recommended while P^C (Carli) is banned.

* referring for example to the same commodity in different outlets

d) Stochastic and aggregative approach to index theory

	(old*) stochastic approach	aggregative approach*
notion of the price level	objective, inflation as the result of monetary factors (equally influencing all prices)	subjective, i.e. defined with reference to the expenditures ("baskets") of specific consuming units (e.g. households)
preferred type of index	**unweighted means of price relatives** (index conceived as [arithmetic, geometric etc] mean of a distribution of relatives)	derived from comparing expenditures (aggregates) in period t with those in the base period 0, consumer price index as a **ratio of expenditures** (of households)

* as opposed to new (see sec. 2.3), some authors distinguish *unweighted* (= old) and *weighted* (= new)

The formulas of Laspeyres and Paasche for example, can be interpreted in *both* ways, as (weighted) means of price relatives (in line with the stochastic approach) and as ratio of expenditures (as required in the aggregative approach). It should be noticed that *none* of the interpretations applies to the "ideal" index of Fisher or to all sorts of chain indices.

1.3. Index formulas of Laspeyres and Paasche

a) Dutot's index and Drobisch's unit value index	c) Commensurability and time reversal test
b) Carli's index formula	d) Stochastic and aggregative approach

a) Price indices, dual interpretation

(1.3.1) $$V_{0t} = \frac{\Sigma p_{it} q_{it}}{\Sigma p_{i0} q_{i0}} = \frac{\Sigma p_t q_t}{\Sigma p_0 q_0} = \frac{\mathbf{p}_t' \mathbf{q}_t}{\mathbf{p}_0' \mathbf{q}_0}$$ (value index).

(1.3.2) $$P_{0t} = \frac{\Sigma p_{it} q_i}{\Sigma p_{i0} q_i} = \frac{\Sigma p_t q}{\Sigma p_0 q} = \frac{\mathbf{p}_t ' \mathbf{q}}{\mathbf{p}_0 ' \mathbf{q}}.$$

(1.3.2a) $$P_{0t} = \Sigma \frac{p_t}{p_0} \cdot \left(\frac{p_0 q}{\Sigma p_0 q} \right) = \Sigma a_{0t}^i w_i$$ (fixed budget index)

(1.3.3) $P_{0t}^{L} = \frac{\Sigma p_t q_0}{\Sigma p_0 q_0}$ **Laspeyres** (1864) (1.3.4) $P_{0t}^{P} = \frac{\sum p_t q_t}{\sum p_0 q_t}$ **Paasche** (1874)

Both price indices, Laspeyres and Paasche can be interpreted in terms of "changing costs of a budget (basket)". It should be noted, however, that there is not a complete symmetry*:
period 0 denotes a *single*, **constant** (for the "life time" of an index) period while
period t is a *variable* period referring to **many** different periods, t = 1, 2 ... .

* A symmetric interpretation of the Laspeyres and Paasche formula is also very popular in the case of the so-called "economic theory" of index numbers.

(1.3.3a) $P_{0t}^{L} = \sum \frac{p_t}{p_0} \left(\frac{p_0 q_0}{\sum p_0 q_0} \right)$ Laspeyres index mean-of-relatives-form.

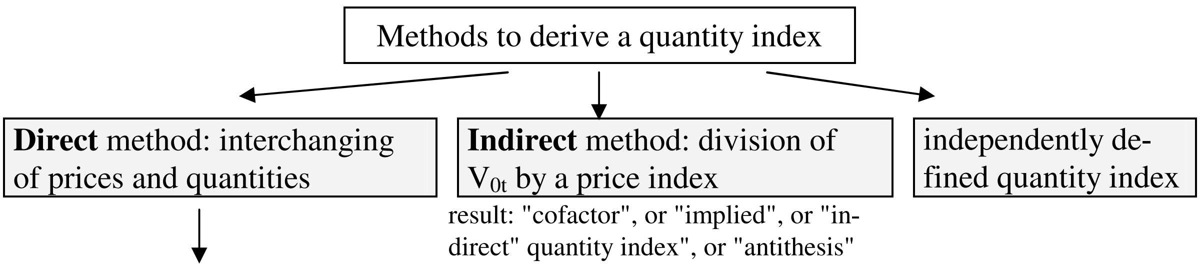
(1.3.4a) $\left(P_{0t}^{P}\right)^{-1} = \sum \frac{p_{i0}}{p_{it}} \frac{p_{it} q_{it}}{\sum p_{it} q_{it}}$ Paasche index mean-of-relatives-form.

b) Price indices and quantity indices

Direct method by *interchanging prices and quantities* in the aggregative form of a price index $Q = f(\mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_t, \mathbf{p}_t)$ from $P = f(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$.

(1.3.5) $Q_{0t}^{L} = \sum b_{0t}^{i} \; w_i$ ($w_i = p_{i0}q_{i0}/\Sigma p_{i0}q_{i0}$)

Figure 1.3.1: Price and quantity indices



index	Laspeyres index	Paasche index
price	$P_{0t}^{L} = \frac{\sum p_t q_0}{\sum p_0 q_0} = \frac{\mathbf{p}_t' \mathbf{q}_0}{\mathbf{p}_0' \mathbf{q}_0}$	$P_{0t}^{P} = \frac{\sum p_t q_t}{\sum p_0 q_t} = \frac{\mathbf{p}_t' \mathbf{q}_t}{\mathbf{p}_0' \mathbf{q}_t}$
quantity	$Q_{0t}^{L} = \frac{\sum q_t p_0}{\sum q_0 p_0} = \frac{\mathbf{q}_t' \mathbf{p}_0}{\mathbf{q}_0' \mathbf{p}_0}$	$Q_{0t}^{P} = \frac{\sum q_t p_t}{\sum q_0 p_t} = \frac{\mathbf{q}_t' \mathbf{p}_t}{\mathbf{q}_0' \mathbf{p}_t}$

The formulas of Laspeyres and Paasche are related to the value index in the following manner

(1.3.6) $V_{0t} = P_{0t}^{L} Q_{0t}^{P} = P_{0t}^{P} Q_{0t}^{L}$

showing that Q_{0t}^{P} is the cofactor (or "factor antithesis") of P_{0t}^{L} and so is Q_{0t}^{L} to P_{0t}^{P}.

c) Asymmetry in the interpretation of the Laspeyres and Paasche formula

There is a remarkable difference between the two formulas, in particular with respect to:

- data requirements,
- an interpretation in terms of representativity and pure price comparison, and

- the underlying concept of measuring a price movement (rise or decline of prices).

Sequence of index numbers

$$P_{01}^{L}, P_{02}^{L}, P_{03}^{L} \ldots : \quad \frac{\sum p_1 q_0}{\sum p_0 q_0}, \frac{\sum p_2 q_0}{\sum p_0 q_0}, \frac{\sum p_3 q_0}{\sum p_0 q_0}, \ldots$$

$$P_{01}^{P}, P_{02}^{P}, P_{03}^{P} \ldots : \quad \frac{\sum p_1 q_1}{\sum p_0 q_1}, \frac{\sum p_2 q_2}{\sum p_0 q_2}, \frac{\sum p_3 q_3}{\sum p_0 q_3}$$

The same consideration applies to series of successive Laspeyres- and Paasche quantity indices (which may result from deflation with P_{0t}^{P}): The **Laspeyres** indices (of prices or quantities) provide a result fully in line with the spirit of the principle of **pure comparison**, successive values are influenced ***only*** by the variable in question, that is prices or quantities respectively. Paasche indices, however, are always affected by ***both*** variables.

The following "antithetic" relationship was known already to Irving Fisher

(1.3.7) $P_{t0}^{L} = 1/P_{0t}^{P}$ and (1.3.8) $P_{t0}^{P} = 1/P_{0t}^{L}$ ("time antithesis").

(1.3.7a) $Q_{t0}^{L} = 1/Q_{0t}^{P}$ and (1.3.8a) $Q_{t0}^{P} = 1/Q_{0t}^{L}$.

(1.3.9) $$P_{0t}^{F} P_{t0}^{F} = \sqrt{P_{0t}^{L} P_{0t}^{P}} \sqrt{P_{t0}^{L} P_{t0}^{P}} = \sqrt{P_{0t}^{L} P_{0t}^{P}} \sqrt{\left(1/P_{0t}^{P}\right)\left(1/P_{0t}^{L}\right)} = 1 .$$

Table 1.3.1: Interpretation of the Laspeyres and Paasche price and quantity index formula

	Price-index formula	
	Laspeyres	Paasche[1)]
numerator	**imputed** expenditures, i.e. expenditures as they *were*, if quantities were kept constant	**empirical**, i.e. observed actual (current period) expenditures referring to actual quantities (*not* to constant quantities)
denominator	**empirical**, i.e. actual base period expenditures (being constant); empirically observed and constant	**imputed** expenditures as they *were*, if prices were kept constant; measures of volume (as substitute for quantity)
time series	time series interpretation possible because only the numerator varies in $P_{01}^{L}, P_{02}^{L}, P_{03}^{L} \ldots$	both, numerator and denominator vary, indices in a "run" $P_{01}^{P}, P_{02}^{P}, P_{03}^{P} \ldots$, not comparable
price movement	**directly** measured: rising (descending) prices inferred from rising (decreasing) costs of a fixed budget[2)]	**indirectly**[2)] measured: rising prices because actual costs are higher as they were when prices remained constant
	Quantity-index formula	
concept of quantity movement	**direct**: quantities (volume) increased to the extent to which expenditure valued at constant prices has increased (i.e. rising volume)	**indirect**[3)]: quantities are rising if value at current prices $\Sigma p_t q_t$ is greater than $\Sigma p_t q_0$.

1 It is assumed that the Paasche price (quantity) index is used to measure price (quantity) *level* movement, but the real merits of the Paasche formula can be seen only in the case of deflation (see **sec. 5.2**).

2 a quantitatively fixed budget

3 indirect approach means: by comparing value and volume (both referring to actual consumption quantities).

d) Theorem of Ladislaus von Bortkiewicz concerning the relationship between Paasche and Laspeyres formulas

Denoting the price and quantity relative of an individual commodity i by a_{0t}^i and b_{0t}^i we obtain the value ratio (relative) $c_{0t}^i = a_{0t}^i \cdot b_{0t}^i$ and by using weights $w_i = p_{i0}q_{i0} / \sum p_{i0}q_{i0}$

(1.3.10) $P_{0t}^L = \sum_i a_{0t}^i w_i$ (1.3.11) $Q_{0t}^L = \sum_i b_{0t}^i w_i$ (1.3.12) $V_{0t} = \sum_i a_{0t}^i b_{0t}^i w_i$

The covariance C between (weighted) price and quantity relatives is given by

(1.3.12) $$C = \sum_i (a_{0t}^i - P_{0t}^L)(b_{0t}^i - Q_{0t}^L)\, w_i = V_{0t} - P_{0t}^L Q_{0t}^L = Q_{0t}^L (P_{0t}^P - P_{0t}^L) = P_{0t}^L (Q_{0t}^P - Q_{0t}^L)\,,$$

or using r_{ab}, the correlation coefficient, and V_a, V_b the coefficients of variation, we get

(1.3.13) $$C = V_{0t} - P_{0t}^L Q_{0t}^L = r_{ab}\left(s_a s_b\right) = r_{ab} P^L Q^L V_a V_b$$

which is known as theorem of Bortkiewicz. It is often stated as follows

(1.3.14) $$\frac{V_{0t}}{P_{0t}^L Q_{0t}^L} = 1 + r_{ab} V_a V_b\,.$$

(1.3.15) $$\frac{P_{0t}^L}{P_{0t}^P} = \frac{Q_{0t}^L}{Q_{0t}^P} = 1 - \frac{C}{V_{0t}} = \frac{P_{0t}^L Q_{0t}^L}{V_{0t}}$$ and (1.3.15a) $$\frac{P_{0t}^P}{P_{0t}^L} = \frac{Q_{0t}^P}{Q_{0t}^L} = 1 + r_{ab} V_a V_b\,,$$

Table 1.3.2: Relations between Laspeyres- and Paasche formulas

	Paasche	Laspeyres
$P_{0t}Q_{0t}$	$P^P Q^P = V^2/(V-C) < V$ if $C < 0$, $> V$ if $C > 0$	$P^L Q^L = V - C > V$ if $C < 0$, $< V$ if $C > 0$
$P_{0t}P_{t0}$	$P_{0t}^P P_{t0}^P = P_{0t}^P / P_{0t}^L < 1$ if $C < 0$	$P_{0t}^L P_{t0}^L = P_{0t}^L / P_{0t}^P > 1$ if $C < 0$

The so called "Laspeyres-effect", that is the situation in which $P^L > P^P$ (and consequently also $Q^L > Q^P$) occurs when the price of commodity i rises (that is $a_{0t}^i > 1$) the quantity tends to be reduced ($q_{it} < q_{i0}$ hence $b_{0t}^i < 1$) and vice versa, that is we have a negative covariance C.

Digression:

Critique of the Laspeyres index and the principle of "pure price comparison" by the US-Senate Advisory Commission (Boskin Commission, BC)

The Laspeyres approach to price level measurement has often been criticized because of its fixed basket. There is a widespread belief that a "fixed basket" index overstates inflation (is "biased" upwards) and that a chain index, or any other index giving a higher weight to the more recent consumption pattern will do a better job and will result in a lower inflation rate. Furthermore advocates of the "*economic theory"* (or true cost of living index = COLI) approach also heavily criticize the Laspeyres principle because the point of reference should not be the same quantity of products in period t and 0 but rather the same utility.

Such ideas were vigorously advanced by an Advisory Commission (also known as Boskin Commission [BC] because it was headed by Michael Boskin) to the US-Senate Commission of Finance[1] to

[1] It is worth noticing that the installation of the BC should be seen against the background of making extensively use of indexation and thus having a problem with budget and debt management in the USA, calling for a cut in

explore a possible upwards bias of the US Consumer Price Index (CPI, then a traditional fixed basket Laspeyres index). The BC published an interim report in 1995 and a final report in 1996 titled "Toward a More Accurate Measure of the Cost of Living"[2] - both widely made public in the internet - from which subsequently some quotations are taken.

Table 1.3.3: Examples for Boskin Commission's suggestions for improvement of the CPI

	price quotations in traditional CPI	Boskin Commission's proposal
medical care*	expensive surgical operations in the case of heart attacks or ulcer	treating heart attacks or ulcer with generic drugs
entertain-ment	visits to cinemas and theatres buying old-fashioned bound-book versions of encyclopaedias	rent of videos, seeing a movie at home CD-ROM encyclopaedias, surfing the internet, going to libraries

* Interestingly substitutions to maintain a constant level of *utility* may occur across borders of a commodity classification: for example services (surgical operations) are substituted by goods (generic drugs), and vice versa (buying books vs. renting books).

Critique of the "utility" reasoning in the COLI-approach:

> 1. The distinction between inflation and welfare measurement becomes blurred, questionable imputations of gains or losses in utility are instigated, and 2. the notion of "good" becomes boundless, and finally 3. we move away from statistics of observable phenomena to speculations about levels of utility or a "fair" amount of income necessary for a "compensation".

Figure 1.3.2: Conceptual differences in price level measurement

Price index should measure according to

	the Laspeyres' formula = **COGI**	the Boskin Commission = **COLI**
the general concept	the change of **expenditure** needed to **buy a** constant (with respect to kind and quantities of goods) "**basket**"	change of average **costs of** a unit of "**utility**" U (on the basis of observed purchases and imputed effects on U)
objective of measurement	"pure" comparison of prices of a well defined set of commodities	changing (due to prices) cost of living at a constant level of U (or wellbeing)
why adjustment for quality change?	to ensure comparability of prices over time (to isolate a genuine increase from a rise due to quality improvement)	to infer gains in utility (U) incorporated in new goods (making maintaining a given level of U cheaper: more U for the same price)

expenditures for social purposes in these days. Hence there was most obviously a certain political interest in getting a lower rate of inflation. It is interesting (and depressing) to note that given this political background there were many theoreticians and statisticians who readily agreed in a unanimous critique of the Laspeyres formula as overstating inflation and who worked out estimates (then sometimes called "guestimates") of the amount of bias in the US-CPI. On the other hand those defending Laspeyres' index formula were few if any.

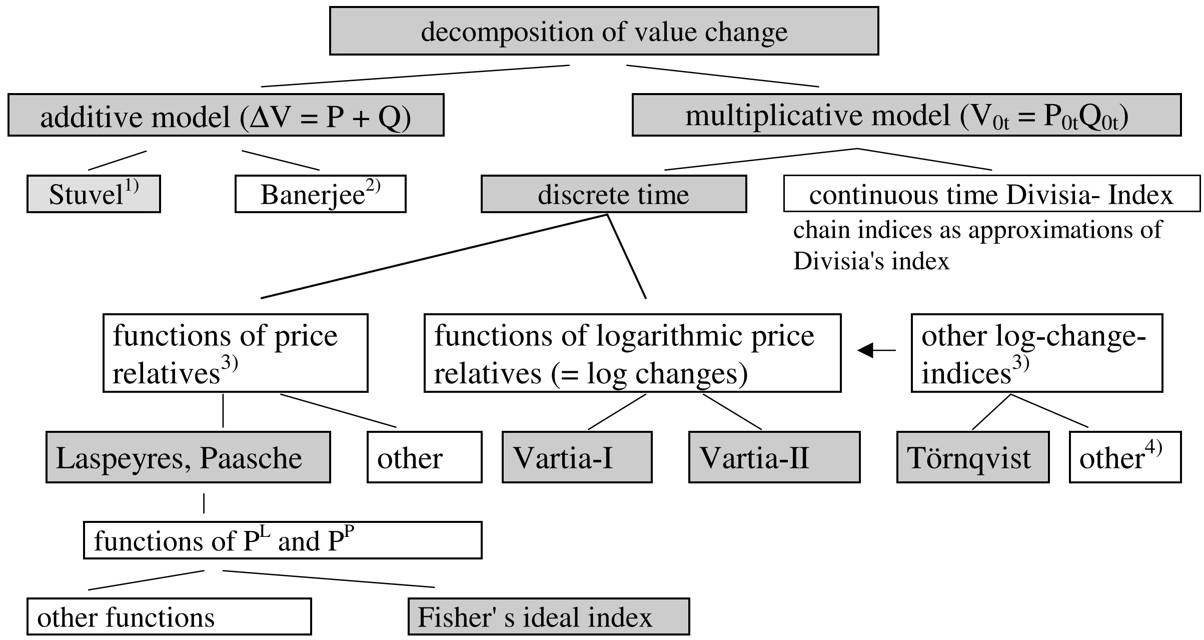
[2] The title was misleading because the concern was a **new** conceptual basis of measurement, rather than how to observe and measure more accurately price movement on the basis of a **given** generally accepted way to measure inflation.

With a little imagination we will always find some additional sources of utility. The US-CPI had been criticized by the BC for example for not taking into account "the benefit to consumers of being able to keep track more easily of children, spouses, or of aged parents" in reporting prices of cellular telephones. As if this kind of benefits has to be seen as equivalent to offering telephones at a lower price.

Chapter 2 Approaches to index theory

2.1. Outline of index theories/approaches

Figure 2.1.1: Some "constructive" approaches in index theory



1) shading of boxes indicates that this index is an "ideal index" (i.e. satisfying the factor reversal test).
2) factorial approach.
3) these indices are not derived from a model of decomposing the value change.
4) as an example: Cobb-Douglas index (transitivity holds) or refined Törnqvist approaches in order to come closer to factor reversibility (Theil, Sato).

2.2. Irving Fisher's mechanistic approach and reversal tests

a) Fisher's systematic search for formulas	d) A weak variant of the time reversal test
b) Generalization of means	e) Fisher's philosophy, the circular test
c) Fisher's reversal tests, crossing of formulas	

a) Fisher's systematic search for formulas

Fisher introduced *four* methods of *weighting* price relatives:

I	base weights: p_0q_0 (Laspeyres)	III	pure price movement: p_tq_0 "hybrid"
II	real expenditures: p_0q_t "hybrid"	IV	current weights: p_tq_t (Paasche)

and he combined them with six types of means. The result is a collection of "second generation" index-formulas such as P^L, P^P, P^{AH}, P^{PA}, P^{HB}, and P^{HH} (**fig. 2.2.1**).

Figure 2.2.1: Family tree of index formulas (according to Köves)

First generation indices (unweighted indices)

Fisher's system of six types of means				
additive structure			geometric block see tab.2.2.1	positional parameters
1	2	3	4	5, 6
aggregative	arithmetic	harmonic	geometric mean	median and mode
Dutot P_{0t}^{D}	Carli P_{0t}^{C}	P_{0t}^{HM}	Jevons P_{0t}^{JV}	

Second generation indices (weighted indices)

means	I: base p_0q_0	II: hybrid p_0q_t	III: hybrid p_tq_0	IV: current p_tq_t
	Fisher's four types of weights			
arithmetic	P_{0t}^{L} Laspeyres	P_{0t}^{P} Paasche	P_{0t}^{AH}	P_{0t}^{PA} Palgrave
harmonic	P_{0t}^{HB} see eq. 2.2.2	P_{0t}^{HH}	P_{0t}^{L} Laspeyres	P_{0t}^{P} Paasche
geometric	less well known formulas, see text below and table 2.2.1			

Note that some of the combinations are identical due to inherent relations between the arithmetic and the harmonic mean: both index formulas, P^L and P^P can be expressed in two ways, using pure and hybrid weights.

Third generation indices see tab. 2.2.1 (derived by crossing)

crossing* (averaging) of	
preferred mean: arithmetic, harmonic, geometric	
formulas (preferred: P_{0t}^{L}, P_{0t}^{P})	**weights** (preferred I, IV)
formulas of Drobisch, Fisher etc.	formulas of Marshall/Edgeworth, Walsh, etc.

* "Crossing" in a more specific use refers to averaging an index P with its "antithesis". If this is done using a *geometric* mean Fisher called it "rectifying".

(2.2.1) $$P_{0t}^{PA} = \sum \frac{p_t}{p_0} \frac{p_t q_t}{\sum p_t q_t} = \frac{\sum p_t \tilde{q}_t}{\sum p_t q_t} > P_{0t}^{P} \text{ where } \tilde{q}_t = q_t \frac{p_t}{p_0}$$ (R. H. I. Palgrave, 1886)

(2.2.2) $$P_{0t}^{HB} = \frac{\sum p_0 q_0}{\sum (p_0^2 q_0 / p_t)} \leq P_{0t}^{L}.$$

(2.2.3) $$P_{0t}^{CD} = \prod \left(\frac{p_t}{p_0}\right)^{\alpha_i} = \prod \left(a_{0t}^{i}\right)^{\alpha_i}$$ (CD = Cobb-Douglas)

The name refers to the well known Cobb-Douglas (production) function.

where the α_i-coefficients are any real valued arbitrary weights not further specified, except for $\alpha_i \geq 0$ and $\Sigma\alpha_i = 1$. Consider two factors, x_1, x_2 only. It is easy to see that the geometric mean of these values, that is $\bar{x}_G = \sqrt{x_1 \cdot x_2}$ will be the geometric mean of the arithmetic and the harmonic mean, thus

(2.2.4) $\bar{x}_G = \sqrt{\bar{x} \cdot \bar{x}_H}$ which is the reason for the relation between P^F, P^{HPL} and P^{DR}.

The harmonic mean of quantities is defined as $q_{iH} = \frac{2(q_{i0}q_{it})}{q_{i0} + q_{it}}$ (Geary-Kahmis method, → ch. 8).

Table 2.2.1: Some "log-change-indices" derived from Fisher's scheme to systematize index formulas, geometric means and types of weights

weights I: $w_{i0} = p_0 q_0 / \sum p_0 q_0$	weights IV: $w_{it} = p_t q_t / \sum p_t q_t$
logarithmic Laspeyres index	logarithmic Paasche* index
$DP^L_{0t} = \prod \left(\frac{p_{it}}{p_{i0}} \right)^{w_{i0}}$	$DP^P_{0t} = \prod \left(\frac{p_{it}}{p_{i0}} \right)^{w_{it}}$

geometric mean of both indices: Törnqvist index
$P^T_{0t} = \sqrt{DP^L_{0t} DP^P_{0t}} = \prod (p_t / p_0)^{\bar{w}}$ where $\bar{w} = \frac{1}{2}(w_0 + w_t)$ or $\ln(P^T_{0t}) = \frac{1}{2}\left[\ln(DP^L_{0t}) + \ln(DP^P_{0t})\right]$
There are similarities between P^F and P^L, P^P on the one hand and P^T and DP^L, DP^P on the other hand.

* Obviously the construction of DP^P_{0t} suggests the name "logarithmic Palgrave" rather than "logarithmic Paasche". Yet it is consistent to refer to Paasche in this context as shown in **sec. 3.4**.

P^{ME} makes use of the arithmetic mean of weights, $\frac{1}{2}(q_{i0} + q_{it})$ and it also takes the form of a weighted mean of P^L and P^P: (2.2.8a) $P^{ME}_{0t} = \frac{P^L_{0t} + V_{0t}}{1 + Q^L_{0t}} = \left(\frac{1}{1 + Q^L_{0t}} \right) P^L_{0t} + \left(\frac{Q^L_{0t}}{1 + Q^L_{0t}} \right) P^P_{0t}$.

(2.2.11) $$\frac{\sum p_t q_t}{\sum p_0 q_0} \left[\frac{\sum p_0 q_0 \frac{q_0}{q_t} \sum p_t q_0 \left(\frac{q_0}{q_t} + \frac{p_t}{p_0} \right) \sum p_t q_t \frac{p_t}{p_0} \left(\sum p_0 q_t \right)^2}{\sum p_0 q_0 \sum p_t q_0 \sum p_t q_t \frac{q_t}{q_0} \sum p_0 q_t \left(\frac{q_t}{q_0} + \frac{p_0}{p_t} \right)} \right]^{1/4}$$ Legacy of Irving Fisher

Table 2.2.2: Some well known formulas derived from crossing[1)] formulas and weights

arithmetic mean	geometric mean	harmonic mean
crossing Laspeyres and Paasche index formula		
(2.2.5) Drobisch 1871 (or Sidgwick1883) $P^{DR}_{0t} = \frac{1}{2}(P^L_{0t} + P^P_{0t})$	(2.2.6) Irving Fisher1922 $P^F_{0t} = \sqrt{P^L_{0t} \cdot P^P_{0t}}$ = "ideal index" of Fisher	(2.2.7) no-name index[2)] (not in use), to be called P^{HPL} $P^{HPL}_{0t} = (P^F_{0t})^2 / P^{DR}_{0t}$
crossing Laspeyres and Paasche weights		
(2.2.8) Marshall and Edgeworth 1887 $P^{ME}_{0t} = \frac{\sum p_t (q_0 + q_t)}{\sum p_0 (q_0 + q_t)}$	(2.2.9) Walsh 1901 $P^W_{0t} = \frac{\sum p_t \sqrt{q_0 q_t}}{\sum p_0 \sqrt{q_0 q_t}}$	(2.2.10) Geary-Khamis[3)] $P^{GK}_{0t} = \frac{\sum p_t \frac{q_0 q_t}{q_0 + q_t}}{\sum p_0 \frac{q_0 q_t}{q_0 + q_t}}$

1) As the Paasche formula is the time and factor "antithesis" of Laspeyres and vice versa, we may say, second generation indices are derived from crossing a formula with her antithesis instead of crossing two formulas.

2) The name, given here for convenience of presentation, should be "Harmonic-Paasche-Laspeyres" (HPL).
3) Result of applying the method of Geary and Khamis for multinational comparisons to m = 2 countries only, A and B. The periods 0 and t are taking in eq. 10 the place of the countries A and B.

Figure 2.2.2: Interpretations of second and third generation indices

second generation	Laspeyres (P^L), Paasche (P^P), Palgrave (P^{PA}), log. Lasp. (DP^L)
third generation	Drobisch (P^{DR}), Fisher (P^F), Marshall-Edgeworth (P^{ME}), Walsh (P^W), Geary-Khamis (P^{GK})

Admissible interpretations

mean of individual price relatives	ratio of aggregates (expenditures, revenues)[1]	mean[2] of index formulas P^L and P^P
applies to *all* **second** generation indices[3]	applies to (= yes) P^L, P^P; with reference to q_0, q_t;	P^L: $\alpha = 1$, P^P: $\alpha = 0$, P^{PA} farfetched[4], no: DP^L
no **third** generation index	yes (average quantities): P^{ME}, P^W, P^{GK}, no: P^{DR}, P^F	P^{DR}, P^F yes ($\alpha = ½$), P^{ME} yes; however P^W, P^{GK} no

1) that is numerator and denominator can be expressed (regarded) as sums of expenditures (products of prices and certain quantities q*, not necessarily the same in numerator and denominator).

2) this means: the index function can be expressed as a special case of the general (weighted) arithmetic mean $\alpha P^L + (1-\alpha)P^P$ or of the general (weighted) geometric mean $(P^L)^{\alpha} + (P^P)^{1-\alpha}$
3) interpretation possible by kind of construction of this type of indices (as they are derived as means)
4) quantities in the numerator, $q_t(p_t/p_0)$ might be viewed as "adjusted" quantities q_t. The situation is similar in cases, like P^{AH}, P^{HB} and P^{HH} in fig. 2.2.1.

b) Generalization of means

The unweighted arithmetic mean is a special case of the weighted linear combination (or *weighted* arithmetic mean, or *convex combination*)

(2.2.12) $P_{0t}^{CC}(\alpha) = \alpha\, P^L + (1-\alpha)P^P$ of two indices, P^L and P^P.

Depending on the parameter α we get:

$$P_{0t}^{CC}(\alpha = 0) = P_{0t}^{P} \quad P_{0t}^{CC}(\alpha = \tfrac{1}{2}) = P_{0t}^{DR} \quad P_{0t}^{CC}(\alpha = \left(1+Q^L\right)^{-1}) = P_{0t}^{ME} \quad P_{0t}^{CC}(\alpha = 1) = P_{0t}^{L}$$

The generalized *weighted geometric mean* P_{0t}^{GM}

(2.2.13) $P_{0t}^{GM}(\alpha) = \left(P_{0t}^{L}\right)^{\alpha} \cdot \left(P_{0t}^{P}\right)^{(1-\alpha)}$

known also as ***generalized Fisher index***. Depending on the parameter α we get

$$P_{0t}^{GM}(\alpha = 0) = P_{0t}^{P} \quad P_{0t}^{GM}(\alpha = \tfrac{1}{2}) = P_{0t}^{F} \quad P_{0t}^{GM}(\alpha = 1) = P_{0t}^{L}$$

It follows from above $P_{0t}^{P} \leq P_{0t}^{F} \leq P_{0t}^{L}$, or $P_{0t}^{P} \geq P_{0t}^{F} \geq P_{0t}^{L}$.

Likewise P^{DR} (Drobisch), and P^{HPL} (harmonic mean) will always lie in the interval bounded by P^P and P^L (where normally P^P will be the lower bound and P^L the upper bound).

The generalized (weighted) *harmonic* crossing of formulas is

(2.2.14) $P_{0t}^{H}(\alpha) = \frac{P_{0t}^{L} P_{0t}^{P}}{\alpha P_{0t}^{L} + (1-\alpha) P_{0t}^{P}}, \alpha = \frac{1}{2} \rightarrow P_{0t}^{HPL}$ has no relevance.

$P^{CC}(\alpha)$, and $P^{H}(\alpha)$ respectively (where $\alpha = ½$) both violate the time reversal test and are "time antithesis" of one another, however $P_{t0}^{GM}(\alpha) = \frac{1}{P_{0t}^{GM}(\alpha)}$ and $P_{0t}^{GM}(\alpha) \cdot Q_{0t}^{GM}(\alpha)$ differs from $V_{0t} = P^{L}Q^{P} = P^{P}Q^{L}$ if $\alpha \neq ½$. Hence

All indices of the generalized Fisher index type ($0 < \alpha < 1$, $\alpha \neq 0$ and $\alpha \neq 1$) will • satisfy the time reversal test, but • fail the factor reversal test, except for the "ordinary" Fisher index (e.g. the case $\alpha = ½$).

It can be shown that not only crossing of formulas but also crossing of weights q_0 and q_t leads to formulas with values that lie in the interval $[P^P, P^L]$.

The concept of a ***power mean*** (or *moment mean*) $\bar{x}_p(r)$ of degree r with weights (relative frequencies) $h_1, h_2, ..., h_m$ in statistics is defined as follows:

(2.2.17) $\bar{x}_P(r) = \left[h_1 x_1^r + h_2 x_2^r + ... + h_m x_m^r\right]^{1/r}$, or with weights w_i and price relatives

(2.2.17a) $P_{0t}^{PM}(r) = \left[\sum_i w_i \left(\frac{p_{it}}{p_{i0}}\right)^r\right]^{1/r}$.

Example r = 2

(2.2.18) $P_{0t}^{QM} = P_{0t}^{PM}(r=2) = \sqrt{\sum \frac{p_t^2}{p_0^2} \frac{p_0 q_0}{\sum p_0 q_0}}$ (quadratic mean index).

Products of power means[3] like $\bar{x}_p(r) \cdot \bar{x}_p(k)$ especially if k = -r give

(2.2.19) $P_{0t}^{PP}(r) = \left[\sum_i s_{i0} \left(\frac{p_{it}}{p_{i0}}\right)^{r/2}\right]^{1/r} \cdot \left[\sum_i s_{it} \left(\frac{p_{it}}{p_{i0}}\right)^{-r/2}\right]^{-1/r}$

where $s_{i0} = \frac{p_{i0} q_{i0}}{\sum p_{i0} q_{i0}}$ and $s_{it} = \frac{p_{it} q_{it}}{\sum p_{it} q_{it}}$ are expenditure shares. Again Fisher's ideal index is a special case (r = 2). Another special case is r = 1

(2.2.20) $P_{0t}^{PP}(r=1) = V_{0t} / Q_{0t}^{W}$ where $Q_{0t}^{W} = \frac{\sum q_t \sqrt{p_0 p_t}}{\sum q_0 \sqrt{p_0 p_t}}$

which is the Walsh type cofactor price index (or factor antithesis of the quantity index Q^W).

In the unweighted case, that is $s_{i0} = s_{it} = 1/n$ we get

[3] Strictly speaking the following formula is not a product of power means (because of the exponents r/2 and –r/2 instead of r and –r). The formula (2.2.19) is also known as **quadratic mean** (a term, however, also used for eq. 2.2.18).

Summary table regarding products of power means (quadratic mean)

$P_{0t}^{PP}(r)$	weighted (2.2.19)	unweighted ($s_{i0} = s_{it} = 1/n$)
r = 1	V_{0t}/Q_{0t}^{W} Walsh co-factor price index	Hybrid index $P_{0t}^{HYB} = \frac{\sum\sqrt{p_{it}/p_{i0}}}{\sum\sqrt{p_{i0}/p_{it}}}$
r = 2	Fisher's ideal index P_{0t}^{F}	CSWD index $P_{0t}^{CSWD} = \sqrt{\frac{\sum(p_{it}/p_{i0})}{\sum(p_{i0}/p_{it})}}$

Other means in index theory

(2.2.21) $\mathrm{Ex}(\mathbf{x}) = \mathrm{Ex}(x_1,...,x_n) = \ln\left[\frac{1}{n}\sum\exp(x_i)\right]$

is known as (unweighted) ***exponential mean***. With weights h_i, it reads as follows

(2.2.21a) $\mathrm{Ex}(\mathbf{x}) = \mathrm{Ex}(x_1,...,x_n) = \ln\left[\sum\exp(x_i)h_i\right]$ where $\sum h_i = 1$.

The ***logarithmic mean*** of two variables, x_1 and x_2 (introduced already in sec.1.1) is given by

(2.2.22) $L(x_1,x_2) = \frac{x_1 - x_2}{\ln(x_1/x_2)} = \frac{x_2 - x_1}{\ln(x_2/x_1)}$ (defined for two values {x_1 and x_2} only).

c) Fisher's reversal tests, "crossing" and "rectifying" of formulas

> The motivation to require the much stronger, and highly restrictive factor reversal test instead of the product test is rarely if ever spelled out in detail. It seems to be the desire to do both, inflation measurement and deflation with the help of the same price index.

The **product test** does not require P and Q to have formulas of the same structure.

Fisher used the notion of "*time antithesis*" T(P) of an index P, and "*factor antithesis*" of the quantity index Q (that is P = F(Q) which is a *price* index) and the concept of a *double antithesis* (see **fig. 2.2.3**). For example the Laspeyres formula is the "time antithesis" and also the "factor antithesis" of the Paasche formula and vice versa Another idea of Fisher was to cross (average) a price index P (or a quantity index Q) to one of its three antithesis in order to find a new price index formula. A special relationship exists in the case of a geometric mean crossing (= *rectification*) for *any* P in that

(2.2.26) $P^* = \sqrt{P \cdot T(P)}$ meets time reversal test, $P_{0t}^{*} = 1/P_{t0}^{*}$, and

(2.2.27) $P^* = \sqrt{P \cdot F(Q)}$ along with (2.2.27a) $Q^* = \sqrt{Q \cdot F(P)}$

is factor reversible: $P^*Q^* = V$.

The following relationships hold for the antithesis relation:

1. they are *reflexive* T(T(P)) = P, and F(F(P)) = P, and
2. the *symmetry* between time - and factor antithesis relation T(F(P)) = F(T(P))

Using equation 2.2.27 it is always possible to construct a pair of index formulas P* and Q* conforming with factor reversibility. Often, however, it is difficult enough to find a meaningful interpretation for such index formulas, like P* and Q*. For example the index

(2.2.28) $Q_{0t}^{GK} = \sum q_t \left(\frac{p_0 p_t}{p_0 + p_t} \right) \Big/ \sum q_0 \left(\frac{p_0 p_t}{p_0 + p_t} \right)$ is the direct quantity index of P_{0t}^{GK} and the geometric mean of its factor antithesis and P_{0t}^{GK} is

$$P_{0t}^{*(GK)} = \sqrt{\frac{\sum p_t q_t}{\sum p_0 q_0} \left(\frac{\sum p_t \left(\frac{q_0 q_t}{q_0 + q_t} \right)}{\sum p_0 \left(\frac{q_0 q_t}{q_0 + q_t} \right)} \frac{\sum q_0 \left(\frac{p_0 p_t}{p_0 + p_t} \right)}{\sum q_t \left(\frac{p_0 p_t}{p_0 + p_t} \right)} \right)} .$$

The Q* index derived according to eq. 2.2.27a) is in the case of the GK index

$$Q_{0t}^{*(GK)} = \sqrt{\frac{\sum p_t q_t}{\sum p_0 q_0} \left(\frac{\sum q_t \left(\frac{p_0 p_t}{p_0 + p_t} \right)}{\sum q_0 \left(\frac{p_0 p_t}{p_0 + p_t} \right)} \frac{\sum p_0 \left(\frac{q_0 q_t}{q_0 + q_t} \right)}{\sum p_t \left(\frac{q_0 q_t}{q_0 + q_t} \right)} \right)}$$

obviously $P_{0t}^{*(GK)} Q_{0t}^{*(GK)} = V_{0t}$

$$P_{0t}^{F} = \sqrt{P_{0t}^{L}\, P_{0t}^{P}} = \frac{1}{P_{t0}^{F}} \text{ and } V_{0t} = P_{0t}^{F} Q_{0t}^{F} = P_{0t}^{F} \sqrt{Q_{0t}^{L}\, Q_{0t}^{P}} = \underbrace{\sqrt{P_{0t}^{L}\, Q_{0t}^{P}}}_{\sqrt{V_{0t}}} \underbrace{\sqrt{P_{0t}^{P}\, Q_{0t}^{L}}}_{\sqrt{V_{0t}}} .$$

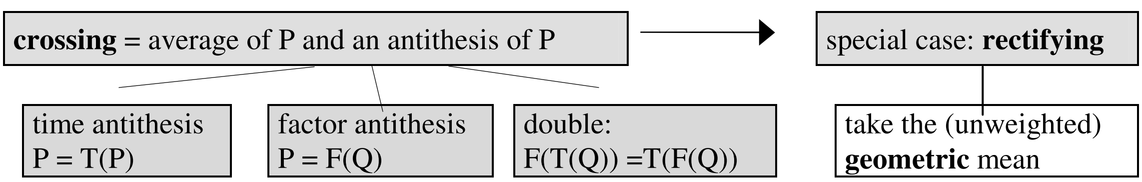
> Index numbers with weights obtained by crossing of weights (e.g. quantity weights q_{i0} and q_{it} in a price index) such as the index formulas of Edgeworth-Marshall (P^{ME}), Walsh (P^{W}) or Geary Kahmis (P^{GK}) always satisfy the *time* reversal test (since the weights are the same whatever the base period may be, 0 or t), but not necessarily the *factor* reversal test.

The same is true for index numbers using any constant weights (the same in numerator and denominator of course) *not depending on periods 0 and t* (like Lowe's prices index, or the Cobb Douglas index).

Crossing of formulas, for example: $P_{0t}^{DR} = \left(P_{0t}^{L} + P_{0t}^{P} \right) \big/ 2$ does not meet the time reversal, and the factor reversal condition either. We get, not surprisingly $P_{t0}^{DR} = \frac{1}{P_{0t}^{HPL}} \neq \frac{1}{P_{t0}^{DR}}$,

showing that the harmonic P^{HPL} - index and the arithmetic Drobisch index are a pair of time antithetic indices like the arithmetic Laspeyres and the harmonic Paasche index.

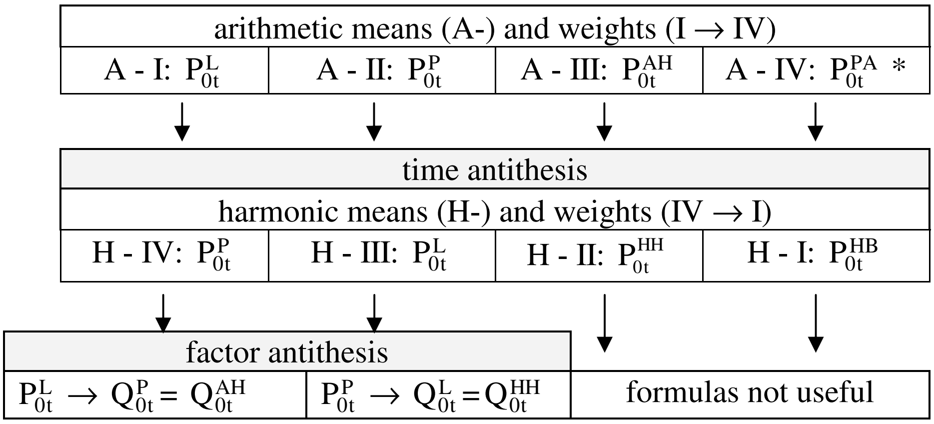
Figure 2.2.3: Crossing P with an antithesis as finder of formulas



time (2.2.23) $P_{0t}^{*} = T(P_{0t}) = 1/P_{t0}$ P* is the time antithesis of P	P^L is $T(P^P)$ since $P_{0t}^{L} = 1/P_{t0}^{P}$ and vice versa P^P = $T(P^L)$
factor (2.2.24) $P_{0t}^{*} = F(Q_{0t}) = V_{0t}/Q_{0t}$ P* is the factor antithesis of Q; accordingly $F(P_{0t})$, the factor antithesis of P_{0t}, is a quantity index	due to $V_{0t} = P_{0t}^{L} Q_{0t}^{P} = P_{0t}^{P} Q_{0t}^{L}$ we have: P^L = $F(Q^P)$ and vice versa $P^P = F(Q^L)$
double (2.2.25) P_{0t}^{*} = F[T(Q)] =T[F(Q)] P* is the double antithesis of Q	P^L is the double antithesis of Q^L, P^P is the double antithesis of Q^P

Figure 2.2.4: Time and factor reversal test of **second** generation indices (additive block only)

weighting schemes: I = p_0q_0 ; II: p_0q_t ; III: p_tq_0 and IV: p_tq_t



* PA = Palgrave

The indices of Geary - Khamis, Walsh and Marshall - Edgeworth pass the time reversal but not the factor reversal test. It can easily be seen that for example indices of Walsh and Marshall - Edgeworth fail the factor reversal test:

$$P^W_{0t} \cdot Q^W_{0t} = \frac{\sum p_t\sqrt{q_0q_t}}{\sum p_0\sqrt{q_0q_t}} \cdot \frac{\sum q_t\sqrt{p_0p_t}}{\sum q_0\sqrt{p_0p_t}} \neq V_{0t}\text{. Likewise}$$

$$P^{ME}_{0t}Q^{ME}_{0t} = V_{0t}\left(1+\frac{1}{Q^P}\right)\left(1+\frac{1}{P^P}\right)\Big/\left(1+Q^L\right)\left(1+P^L\right) \neq V_{0t}.$$

According to the theorem of L. v. Bortkiewicz we have

$$P^L_{0t}P^L_{t0} = \frac{P^LQ^L}{V} = \frac{P^L}{P^P} = 1-\frac{C}{V} \text{ and } P^P_{0t}P^P_{t0} = \frac{P^PQ^P}{V} = \frac{P^P}{P^L}\text{, and we also see that}$$

$$\frac{P^{DR}Q^{DR}}{V} - 1 = \frac{1}{2}\left[\frac{1}{2}\left(\frac{P^LQ^L}{V}-1\right)+\frac{1}{2}\left(\frac{P^PQ^P}{V}-1\right)\right]$$ the relative deviations of P^L and P^P are averaged.

Table 2.2.3: Factor reversal test in the case of **third** generation indices
TR = time reversal test, FR = factor reversal test

name of index	(direct) quantity index	cofactor quantity index	TR	FR
Crossing of **formulas** P^L and P^P				
Drobisch	$Q^{DR} = \frac{1}{2}(Q^L + Q^P)$	$Q^{HPL} = (Q^F)^2 / Q^{DR}$	no	no
Fisher	$Q^F = \sqrt{Q^LQ^P}$	$Q^F = \sqrt{Q^LQ^P}$	yes	yes
harmonic (HPL)	$Q^{HPL} = (Q^F)^2 / Q^{DR}$	$Q^{DR} = \frac{1}{2}(Q^L + Q^P)$	no	no
Crossing of **weights***				
Marhall - Edgeworth	$\frac{1}{1+P^L}Q^L + \frac{P^L}{1+P^L}Q^P$	$\frac{V}{V+P^L} + \frac{P^L}{V+P^L}Q^PQ^L$	yes	no
Walsh	$\frac{\sum q_t\sqrt{p_0p_t}}{\sum q_0\sqrt{p_0p_t}}$	$\frac{\sum p_tq_t}{\sum p_0q_0}\frac{\sum p_0\sqrt{q_0q_t}}{\sum p_t\sqrt{q_0q_t}}$	yes	no

* as for the Geary - Khamis formula see equations 2.2.28 and 28a

d) A weak variant of the time reversal test

It is obviously rather restrictive to require an index P_{t0} to be the inverse index P_{0t}. It appears sufficient to postulate:

(2.2.29) $$\text{if } P_{0t} > 1 \text{ then } P_{t0} < 1 \text{ and if } P_{0t} < 1 \text{ then } P_{t0} > 1.$$

This requirement seems to be reasonable and not too ambitious: it is only desired that an increase in the direction 0 → t should correspond to a decline in the opposite direction t → 0 and vice versa.

Example 2.2.2 Assume the following prices and quantities

i	p_{i0}	p_{it}	q_{i0}	q_{it}
1	12	15	80	20
2	20	18	10	80

Calculate the following indices P^C (Carli), P^L, P^P, P^{DR} (Drobisch), each in both directions, that is 0→t and t→0. The results are as follows:

formula	P_{0t} direction 0 → t	P_{t0} direction t → 0
Carli	$P_{0t}^C = (1.25 + 0.9)/2 = 1.075 > 1$	$P_{t0}^C = 0.9555 < 1$
Laspeyres	$P_{0t}^L = 1380/1160 = 1.1897 > 1$	$P_{t0}^L = 1/P_{0t}^P = 1.0575 > 1$
Paasche	$P_{0t}^P = 1740/1840 = 0.9457 < 1$	$P_{t0}^P = 1/P_{0t}^L = 0.8406 < 1$
Drobisch	$P_{0t}^{DR} = 1.0677 > 1$	$P_{0t}^{DR} = 0.9490 < 1$

Thus both, the Laspeyres- as well as the Paasche formula may fail this weak time reversal test, while the indices of Carli and Drobisch (or Sidgwick) will pass this test *necessarily* (though both indices do *not* satisfy the time reversal test). ♦

e) Fisher's philosophy in evaluating formulas by reversal tests and the circular test

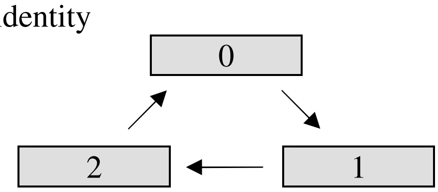
Fisher's ***seven point scale*** ranging from worthless to superlative:

1. worthless, 2. weak, 3. correct, 4. good, 5. very good, 6. excellent and 7. superlative

examples: **5**: Laspeyres and Paasche, **6**: equation 2.2.11, **7**: the two crossed indices P^F and P^{ME}

The meaning and significance of Fisher's circular test

(2.2.30) $$P_{01}\,P_{12} = P_{02}$$, and in connection with identity



(2.2.30a) $$P_{01}\,P_{12}\,P_{20} = P_{00} = 1,$$

(2.2.30b) $$P_{01}\,P_{12}\,P_{23} = P_{03}\,.$$

(2.2.31) $$P_{0t}^{LW} = \frac{\sum p_t q}{\sum p_0 q}$$ (Lowe's price index).

A critique of circularity and time reversibility (Pfouts)

Circularity is tantamount to the requirement that a certain matrix **P** of index numbers has to be *singular*. **P** is defined as follows (in the case of T+1 = 4 rows and columns, t = 0, 1,..., T)

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix}.$$

Fisher's tests, however, tacitly assume **P** being singular. This can easily be seen since in the case of T = 2 we obtain:

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 1 & P_{01} & P_{01}P_{12} \\ 1/P_{01} & 1 & P_{12} \\ 1/P_{01}P_{12} & 1/P_{12} & 1 \end{bmatrix}$$

and the determinant $|\mathbf{P}|$ in fact vanishes. A consequence is that a single additional value, P_{23} is sufficient to calculate a fourth row and column; although we do not even have to know what index formula is being used

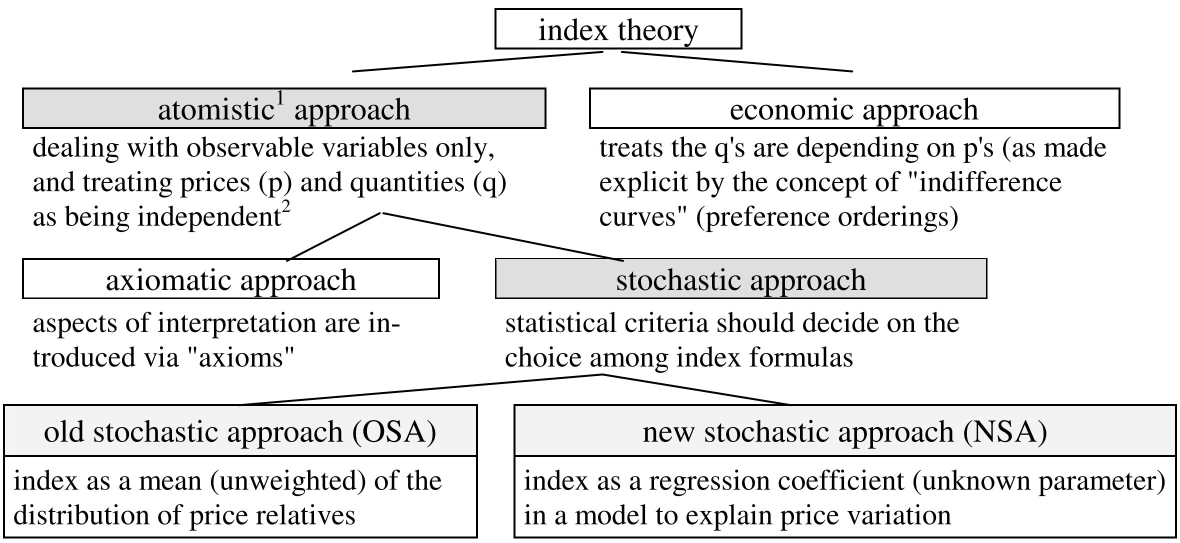
$$\mathbf{Pc} = \begin{bmatrix} 1 & P_{01} & P_{02} \\ 1/P_{01} & 1 & P_{12} \\ 1/P_{01}P_{12} & 1/P_{12} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ P_{23} \end{bmatrix} = \begin{bmatrix} P_{02}P_{23} \\ P_{12}P_{23} \\ P_{23} \end{bmatrix} = \begin{bmatrix} P_{03} \\ P_{13} \\ P_{23} \end{bmatrix} = \mathbf{p}.$$

There are actually only two independent observations, P_{01} and P_{12} assembled in the 3×3 matrix **P**. It can easily be verified that time reversibility implies all second order principal minors of **P** being identically singular, hence determinants like $\begin{vmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{vmatrix}$ or $\begin{vmatrix} P_{33} & P_{35} \\ P_{53} & P_{55} \end{vmatrix}$ will all vanish.

2.3. The stochastic approach in price index theory

The underlying conception of the new stochastic approach (NSA) is to find regression equations in which index formulas are playing the part of regression coefficients. If applied to empirical data the error term in the regression is taken as an indication for the "reliability" or appropriateness of the index function. Goodness of fit is interpreted as caused by the dispersion in the (sample) price data and the index formula in question. The NSA allows to estimate standard errors and confidence intervals of various index formulas and thereby (ostensibly) a better understanding of index formulas.

Figure 2.3.1: The place of the stochastic approach in index theory



1 also called "formal" or "mechanic" (mechanistic) and the like

2 variations in q in response to variations in p are captured indirectly in the NSA since in some regression models expenditure shares are involved

Figure 2.3.2: Some "Budget share weighted average models" (BSW) models

General structure of the model $y_{it} = \theta x_{it} + u_{it}$, where u_{it} is a function of ε_{it} and ε_{it} fulfills the standard assumptions concerning $E(\varepsilon_{it}), V(\varepsilon_{it})$, and $C(\varepsilon_{it}\varepsilon_{jt})$,

index formula[1)]	y-variable	x-variable	error term u	$(n-1)\hat{\sigma}^2_{\hat{\theta}}$ [2)]
Laspeyres	$p^i_{0t}\sqrt{w_{i0}}$	$\sqrt{w_{i0}}$	$\varepsilon_{it}\sqrt{w_{i0}}$	$\sum w_{i0}\left(p^i_{0t} - \hat{\theta}_t\right)^2$
Paasche	$p^i_{0t}\sqrt{w^*_{it}}$	$\sqrt{w^*_{it}}$	$\varepsilon_{it}\sqrt{w^*_{it}}$	$\sum w^*_{it}\left(p^i_{0t} - \hat{\theta}_t\right)^2$
Törnquist $\ln(P^T)$	Dp^i_{0t}		$\varepsilon_{it}/\sqrt{\overline{w}_{it}}$	$\sum \overline{w}_{it}\left(p^i_{0t} - \hat{\theta}_t\right)^2$
Jevons $\ln(P^{JV})$	Dp^i_{0t}		ε_{it} [3)]	$\frac{1}{n}\sum\left(Dp^i_{0t} - \hat{\theta}_t\right)^2$

1) parameter $\hat{\theta}$ equals formula of ...; 2) $\hat{\sigma}^2_{\hat{\theta}}$ denotes the estimated sampling variance of the regression coefficient $\hat{\theta}$; 3) variance $V(\varepsilon) = \sigma^2$

The centerpiece of the NSA is to regard empirical estimates as reflecting the suitability of an index formula. It is our view, however, that this relationship between an index formula on the one hand and the fit of a regression (when applied to data) on the other hand is a misconception.

(2.3.1) $p^i_{0t} = \theta_t + \varepsilon_{it}$,

It is clear that **Carli**'s index (unweighted arithmetic mean of price relatives) will be the least squares (LS) estimator of θ_t in the model of eq. 1, that is $\hat{\theta}_t = \sum p^i_{0t}/n = P^C_{0t}$.

By the same token we get **Jevons**' index (unweighted geometric mean of price relatives)

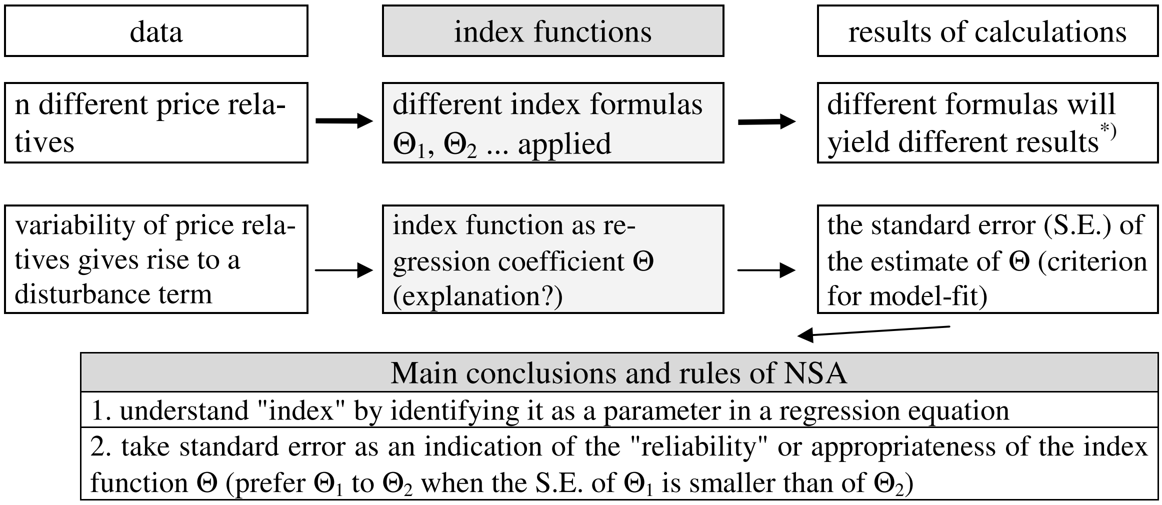
(2.3.2) $Dp^i_{0t} = \ln\left(p^i_{0t}\right) = \theta_t + \varepsilon_{it}$,

Example 2.3.1 The following data for n = 5 commodities are given

i	p_{i0}	p_{it}	q_{i0}	q_{it}	w_{i0}	w_{it}	w^*_{it}	$\overline{w}_{it}$	p^i_{0t}
1	10	12	40	35	0.200	0.1750	0.1392	0.1875	1.200
2	15	20	30	25	0.225	0.2083	0.1491	0.2167	1.333
3	20	15	10	32	0.100	0.2000	0.2545	0.1500	0.750
4	30	25	20	30	0.300	0.3125	0.3579	0.3063	0.833
5	20	20	17.5	12.5	0.175	0.1042	0.0994	0.1396	1.000

index $\hat{\theta}_t$	$(n-1)\hat{\sigma}^2_\varepsilon$	$\hat{\sigma}_\theta = \sqrt{\hat{\sigma}^2_\varepsilon/n}$	bounds of confidence interval	length of conf. interval
Laspeyres $P^L_{0t} = 1.04$	0.0460	0.04795	0.906877; 1.17313	0.266599
Paasche $P^P_{0t} = 0.9543$	0.0459	0.04790	0.82128; 1.0827	0.26599
logarithm of Törnquist $\ln(P^T_{0t}) = -0.00247$	0.0448	0.04835	-0.13671; 0.13177	
retransformed $P^T_{0t} = 0.998$			0.87222; 1.14084	0.26862
$\ln(P^{JV}_{0t}) = 0.0$	0.0460	0.04835	-0.13390; 0.13390	
retransformed $P^{JV}_{0t} = 1$			0.87467; 1.143283	0.268609

Figure 2.3.3: Main ideas of the new stochastic approach



*) It is the essence of the NSA method that the direction of this last arrow can be inverted in that the variability of the results permits an assessment of the index formulas.

2.4. Economic approach (the "true cost of living index", COLI)

The "true cost of living index" (COLI), or "constant utility index" (CU-index) is defined as the ratio of the minimum expenditures required to attain a particular indifference curve (the same utility level) under two price regimes (or price vectors $\mathbf{p}_t$ and $\mathbf{p}_0$), or: it is the amount of income necessary to leave somebody as well off as before the price change. The utility function (graphically represented by the well known "indifference curve") assigns a certain utility level (welfare) U_0 to infinitely many combinations of quantities q_1 and q_2 at base time 0 as follows: $U_0 = U(q_{10}, q_{20})$, the function U and it value U_1 at time 1 being analogously defined. Maximizing the utility U under the constraint of balancing the "budget" (expenditure function y_0) that is

$$\max U(q_{10}, q_{20}) \text{ given } y_0 = p_{10}q_{10} + p_{20}q_{20}$$

has a unique solution (a tangential point of the budget line and the indifference curve) determining the quantities q_{10} and q_{20} and thereby the minimum expenditure $y_0 = y(p_{10}, p_{20}, U_0)$ such that

$$\text{(2.4.1)} \qquad P_{0t}^{CU}(U_0) = \frac{y(\mathbf{p}_t, U_0)}{y(\mathbf{p}_0, U_0)} = \frac{C(t,0)}{C(0,0)}$$

is the "constant utility" (CU) – or "true cost of living" COLI- index. Accordingly the ratio of the minimum expenditures (y), or cost C required to attain the same utility level U_1 is given by

$$\text{(2.4.2)} \qquad P_{0t}^{CU}(U_t) = \frac{y(\mathbf{p}_t, U_t)}{y(\mathbf{p}_0, U_t)} = \frac{C(t,t)}{C(0,t)}.$$

The CU-index depends not only on price vectors but also on the utility level in question, that means that the two indices in eqs. 1 and 2 are not necessarily equal unless in the case of homothetic indifference curves, or (equivalently): a linear homogenous (in the quantities) utility function. Note that

The Laspeyres price index compares expenditures at different price regimes referring to the **same quantities** whilst the CU-index compares expenditures referring to the **same utility**.

$$\text{(2.4.3a)} \qquad P_{0t}^{CU}(U_0) \le P_{0t}^{L} \text{ (upper bound) and lower bound}$$

$$\text{(2.4.3b)} \qquad P_{0t}^{P} \le P_{0t}^{CU}(U_t)$$

provided that indices P^L and P^P refer to a single utility maximizing household "possessing" indifference curve U_0 and U_t respectively.

A system of indifference curves (IC) is called *homothetic* (or *linear homogeneous*) if $U(\lambda \mathbf{q}) = \lambda U(\mathbf{q})$, $\lambda \neq 0$); then each IC is a uniform enlargement, or contraction of each other, and the inequalities 3a/3b can be combined into one single inequality $P_{0t}^{P} \leq P_{0t}^{UC}(U) \leq P_{0t}^{L}$.

Critique of the "utility" reasoning in the COLI-approach:

1. The distinction between inflation and welfare measurement becomes blurred, questionable imputations of gains or losses in utility are instigated, and 2. the notion of "good" becomes boundless, and finally 3. we move away from statistics of observable phenomena to speculations about levels of utility or a "fair" amount of income necessary for a "compensation".

2.5. Chain indices and Divisia's approach (general introduction)

a) Necessary terminological distinctions	c) Divisia index, its relation to chain indices
b) Weights in the chain approach	d) Discrete time approximations and weights

a) Some necessary terminological distinctions

A distinction between "chain" and "fixed based" or "fixed weighted" index is misleading and should be avoided. It is better to distinguish chain- and direct indices.

Figure 2.5.1: Terminological distinctions referring to chain indices

a) Types of comparison between 0 and t

- **direct** index approach: using data of 0 and t only
- **chain** index approach: index defined as a product of links

b) The two elements to define a **chain index**

- constant element: the "**chain**" $\overline{P}_{0t}$, an index to compare 0 and t, gained by multiplication of links
- variable element: definition of a "**link**" P_t^C (index to compare two adjacent periods $P_t^C = P_{t-1,t}$)
  - examples: Laspeyres-, Paasche-, Fisher-type

A chain index essentially is a specific type of temporal aggregation and description of a time series rather than a comparison of two states taken in isolation, it provides a measure of the *cumulated effect* of successive steps (and the shape of the path) from 0 to 1, 1 to 2, ... , t-1 to t.

Three **sources of variation**[4] are responsible for the result in the case of a chain index,

1. the difference in prices in t as compared with 0,
2. the change weights (quantities) have undergone (in response to a change in prices) in comparing 0 and t, and
3. prices and quantities in the intermediate points in time, that is in 1, 2,. .., t-1.

[4] A fourth source is the ever changing "domain of definition" of the index function, which is the often praised ease with which the selection of goods can be changed from one period to another (not only in the case of re-basing).

The *two* "elements" of the definition of chain price[5] indices should be kept distinct:

1. A constant element is the "**chain**" $\overline{P}_{0t}$, which always is a **product** of "links" P_t^C, each of which being a direct index comparing t with the preceding period t-1

   (2.5.1) $\overline{P}_{0t} = P_1^C P_2^C ... P_t^C = \prod_{\tau=1}^{\tau=t} P_\tau^C$, and

2. in defining the **link** $P_t^C = P_{t-1,t}$ there are numerous solutions we might think of[6], giving rise to Laspeyres- , Paasche- Fisher- and other chain index numbers (depending on the type of link P_t^{LC}, P_t^{PC}, P_t^{FC} etc. that are multiplied ["chainlinked"] to get the chain $\overline{P}_{0t}$). The Laspeyres link, as an example is defined as follows

   (2.5.2) $P_t^{LC} = P_{t-1,t}^L = \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}}$, such that $\overline{P}_{0t}^{LC} = P_1^{LC} \cdot ... \cdot P_t^{LC}$ is the Laspeyres chain.

   Since a link always compares the reference period t with the *preceding* base period t-1 there is no need for two subscripts. It is sufficient to use only one subscript, t. The Paasche link obviously is $\Sigma p_t q_t / \Sigma p_{t-1} q_t$.

Due to the multiplication the chain $\overline{P}_{0t}$ is in general a function of the price and quantity vectors $\mathbf{p}_0$, $\mathbf{q}_0$, $\mathbf{p}_1$, $\mathbf{q}_1$, $\mathbf{p}_2$, $\mathbf{q}_2$,...., $\mathbf{p}_{t-1}$, $\mathbf{q}_{t-1}$, $\mathbf{p}_t$, $\mathbf{q}_t$, and not only of the first and last pair of vectors.

Note that the existence of a product representation of an index as such is not sufficient to characterize a chain index

(2.5.3) $\overline{P}_{03}^{LC} = \left(\sum \frac{p_1}{p_0} \frac{p_0 q_0}{\sum p_0 q_0}\right)\left(\sum \frac{p_2}{p_1} \frac{p_1 q_1}{\sum p_1 q_1}\right)\left(\sum \frac{p_3}{p_2} \frac{p_2 q_2}{\sum p_2 q_2}\right)$

(2.5.4) $P_{03}^{L} = \left(\sum \frac{p_1}{p_0} \frac{p_0 q_0}{\sum p_0 q_0}\right)\left(\sum \frac{p_2}{p_1} \frac{p_1 q_0}{\sum p_1 q_0}\right)\left(\sum \frac{p_3}{p_2} \frac{p_2 q_0}{\sum p_2 q_0}\right) = \sum \frac{p_3}{p_0} \frac{p_0 q_0}{\sum p_0 q_0}$.

Note also that the factors on the right hand side (RHS) of eq. 4 are *not* the "ordinary" indices, P_{01}^L, P_{12}^L and P_{23}^L (since P^L is *not* transitive), but a sequence of *rebased* Laspeyres indices

$P_{01} = \frac{P_{01}}{P_{00}} = \frac{\sum p_1 q_0}{\sum p_{01} q_0}$, $P_{12(0)} = \frac{P_{02}}{P_{01}} = \frac{\sum p_2 q_0}{\sum p_1 q_0}$, and $P_{23(0)} = \frac{P_{03}}{P_{02}} = \frac{\sum p_3 q_0}{\sum p_2 q_0}$.

$\overline{P}_{03}^{LC}$ will in general differ from P_{03}^L which is known as *drift* of the chain index[7]. Though *by definition* the following holds

(2.5.5) $\overline{P}_{0t} = \overline{P}_{0k} \overline{P}_{kt}$

The idea of the chain test (or "chainability" or "transitivity") is that the result ($\overline{P}_{0t}$) should be the same for *any* k, irrespective of how the interval (0, t) is partitioned into sub-intervals. But in general this is *not* true in the case of chain indices. Not only is $\overline{P}_{06}^{LC}$ different from P_{06}^L, the case of 1-period-links, $\overline{P}_{06} = P_{01} P_{12} ... P_{56}$ will in general not yield the same result as for example the chaining of 2-period-links $P_{02} P_{24} P_{46}$ (see chapter 7 for more details).

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[5] The definition applies mutatis mutandis also to quantity indices.

[6] The link thus is the variable element of the definition of a chain index.

[7] The term "drift" does not mean that the incorrect chain index is drifting away from the correct direct index. We may of course as well think of the direct index drifting away from the (correct) chain index.

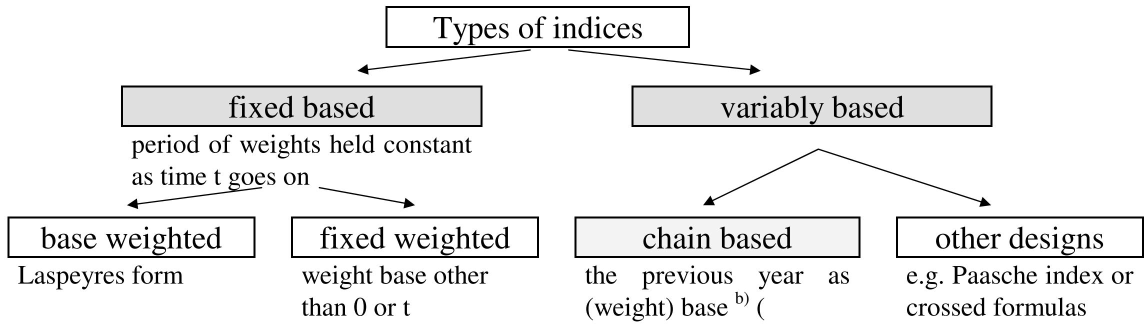
The name "chain index" is misleading: multiplication ("chaining") may give rise to the impression chainability were met. But this is not true. Moreover: Multiplication is not a unique, defining feature of chain indices. Nor are there any desirable properties of chain indices to be concluded from multiplication as such[8].

(2.5.3a) $\overline{P}_{03}^{LC} = \frac{\sum_i p_{1i}q_{0i}}{\sum_i p_{0i}q_{0i}} \frac{\sum_k p_{2k}q_{1k}}{\sum_k p_{1k}q_{1k}} \frac{\sum_m p_{3m}q_{2m}}{\sum_m p_{2m}q_{2m}}$ (Changes in the domains of definition),

b) Weights in the chain approach

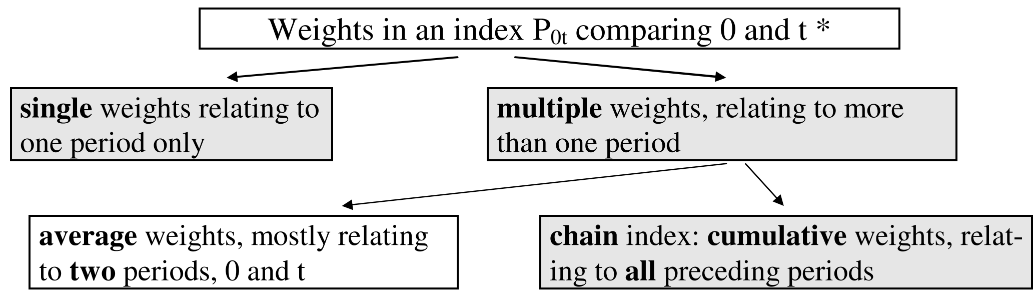
The typology of the following **fig. 2.5.2** is obscuring and we prefer the typology of **fig. 2.5.3**.

Figure 2.5.2: Traditional classification of weighting schemes [a)]



a) Other possibilities exist. The weight base can be a period other than 0 or t in the interval (0,t) or it can be periodic (e.g. each starting point of a business cycle)..
b) this, however, applies only to the links

Figure 2.5.3: Alternative principles to define weights



* This scheme compares **direct** indices with **chains**, not direct indices with **links** (as **fig. 2.5.2** does). A direct index will in general be single-weighted, like P^L or P^P, or average weighted, like P^{ME}, P^T or P^W, whereas a chain index *always* has cumulative weights.

c) Divisia index and its relation to chain indices

It is assumed that two functions, $p_i(t)$ and $q_i(t)$ exist *for each commodity* (i = 1, ... , n) at *any* point in time (a continuous variable). Let $P(\tau)$ denote the (unknown, by contrast to $p_i(t)$) price *level* function (absolute aggregative level) varying continuously over time and let $Q(\tau)$ denote the quantity level defined analogously. It is a mere matter of definition that a value function V (τ) exists as follows

(2.5.6) $V(\tau) = \sum_{i=1}^{n} p_i(\tau)q_i(\tau)$,

[8] On the contrary: to study figures resulting from a complicated mix of influences makes, in general less sense.

(2.5.7) $V(\tau) = P(\tau)\, Q(\tau)$.

Unlike the functions $p_i(\tau)$ and $q_i(\tau)$ the levels P(τ) and Q(τ) are unobservable. They will lead eventually to a "price index" and "quantity index" respectively. Eq. 7 only *defines* the "levels" P(τ) and Q(τ) implicitly by stipulating a relation between V(τ), P(τ) and Q(τ).

Consider differential changes of V(τ) according to eq. 6

(2.5.8) $dV(\tau) = \sum_i q_i(\tau)\, dp_i(\tau) + \sum_i p_i(\tau)\, dq_i(\tau)$.

Dividing both sides by V(t) as given in eq. 6 leads to

$$\frac{dV(\tau)}{V(\tau)} = \frac{\sum q_i(\tau)\, dp_i(\tau)}{\sum q_i(\tau) p_i(\tau)} + \frac{\sum p_i(\tau)\, dq_i(\tau)}{\sum p_i(\tau) q_i(\tau)}, \text{ and}$$

(2.5.9) $$\frac{dV(\tau)/d\tau}{V(\tau)} = \frac{\sum q_i(\tau)\, dp_i(\tau)/d\tau}{\sum q_i(\tau) p_i(\tau)} + \frac{\sum p_i(\tau)\, dq_i(\tau)/d\tau}{\sum p_i(\tau) q_i(\tau)}.$$

(2.5.8a) $$\frac{dV(\tau)}{d\tau} = Q\frac{dP(\tau)}{d\tau} + P\frac{dQ(\tau)}{d\tau} \text{ or}$$

(2.5.9a) $$\frac{dV(\tau)/d\tau}{V(\tau)} = \frac{dP(\tau)/d\tau}{P(\tau)} + \frac{dQ(\tau)/d\tau}{Q(\tau)}.$$

The (continuous time) growth rate of V is the sum of the growth rate of the price level (P) and the quantity level (Q) respectively. Taking the growth rate of P for example we see that it is a weighted average of the growth rates of the prices p_i of the individual commodities (i = 1, …, n)

(2.5.10) $$\frac{dP(\tau)/d\tau}{P(\tau)} = \frac{d\ \ln\ P(\tau)}{d\tau} = \frac{\sum_i q_i(\tau)}{\sum_i q_i(\tau) p_i(\tau)} dp_i(\tau)/d\tau,$$

$$= \sum w_i(\tau)\frac{dp_i(\tau)/d\tau}{p_i(\tau)} = \sum w_i(\tau)\frac{d \ln p_i(\tau)}{d\tau}$$

where weights $w_i(\tau) = p_i(\tau) q_i(\tau) / \sum p_i(\tau) q_i(\tau)$ are expenditure shares at point τ (and hence changing with time) and summation takes place over n commodities. In the same manner the growth rate of Q(τ) is a weighted arithmetic mean of growth rates of n functions $q_i(\tau)$

(2.5.11) $$\frac{dQ(\tau)/d\tau}{Q(\tau)} = \frac{d\ \ln\ Q(\tau)}{d\tau} = \sum w_i(\tau)\frac{d \ln q_i(\tau)}{d\tau}.$$

Justification to identify P(τ) as price level at time τ (and P(t)/P(0) as price index):

Assume quantities in eq. 9 don't change such that $dq_i(\tau) = 0$ for all i then
- the change of the quantity level (that is of Q(τ)) according to eq. 11 should also be zero, or in other words
- the change of volume (in eq. 9) should equal the change of prices.

This applies mutatis mutandis for the assumption of no change of prices $dp_i(\tau) = 0$.

It is this consideration that allows to separate the two differentials (in prices and in quantities) and to identify them as growth rate of price and quantity level respectively.

Another way of looking at Divisia's method is indicated in eq. 10 and 11: a (continuous time) growth rate of the (absolute) price level P(τ), or quantity level Q(τ) is constructed as a weighted sum of n growth rates; weights $w_i(\tau)$ being shares in total value and varying continuously with time. To get a price index the differential equation (eq. 10) is to solve ("integrate") for P

(2.5.12) $$P(t)=P(0)\exp\left(\int_0^t \frac{\sum q_i(\tau)\frac{dp_i(\tau)}{d\tau}}{\sum q_i(\tau)p_i(\tau)}d\tau\right)=P(0)\exp\left(\int_0^t \sum w_i(\tau)\frac{d\ln p_i(\tau)}{d\tau}d\tau\right),$$

and thus the price index is given by

(2.5.12a) $$P_{0t}^{Div}=\frac{P(t)}{P(0)}.$$

Correspondingly Divisia's quantity index is

(2.5.13) $$Q_{0t}^{Div}=\frac{Q(t)}{Q(0)}=\exp\left(\int_0^t \frac{\sum p_i(\tau)\ dq_i(\tau)}{\sum p_i(\tau)q_i(\tau)}\right).$$

The name "integral index" stems from the fact that the pair of Divisia indices is derived by solving (integrating) differential equations. The problem, however, is that the integration suffers from *lack of path invariance:* the solutions (integral functions) of the "line integrals" in eq. 13 and 14 depend on the path connecting 0 and t. By contrast the integration

(2.5.14) $$V_{0t}=\frac{V(t)}{V(0)}=\exp\left(\int_0^t \frac{dV(\tau)/d\tau}{V(\tau)}d\tau\right)=\exp\left(\int_0^t \frac{dV(\tau)}{V(\tau)}\right)=\exp[f(0,t)]$$

depends on the endpoints 0 and t only, not on the (shape of the) path connecting them. Thus

(2.5.14a) $$V_{0t}=V_{0k}V_{kt}=\exp\left(\int_0^k \frac{dV}{V}+\int_k^t \frac{dV}{V}\right)=\exp\left(\int_0^t \frac{dV}{V}\right)$$ or f(0,t) = f(0,k) + f(k,s).

By contrast to f( ) the corresponding functions in eq. 12 and 13 are *not* path invariant.[9]

Not only chain indices but also certain direct indices can be derived

- under specific assumptions concerning the functions $p_i(\tau)$ and $q_i(\tau)$, or
- from various types of discrete time approximations to the continuous time Divisia index.

Consider in τ the same (or for all i proportional) quantities as compared with 0, or assume a (most unlikely) path of quantities such that $q_i(\tau) = \lambda q_i(0) = \lambda q_0$. This gives

(2.5.15) $$\frac{dP(\tau)}{P(\tau)}=\frac{\sum q(\tau)dp(\tau)}{\sum q(\tau)p(\tau)}=\frac{\sum \lambda q_0 dp(\tau)}{\sum \lambda q_0 p(\tau)}=\frac{d\sum q_0 p(\tau)}{\sum q_0 p(\tau)},$$

where the subscript i denoting the commodity is deleted for convenience. Integration of the price differential $\int_0^t \frac{dP(\tau)}{P(\tau)}d\tau$ subject to constant or proportional quantities[10] leads to

(2.5.16) $$\ln P(t)=\ln\left[\sum q_0 p_t\right]+C \text{ where } p_t=p(\tau=t)$$

with an arbitrary constant C, the value of which can be determined by assuming prices p_0 to enter P(0) the price level in base period 0 as do prices p_t with P(t). This means

(2.5.16a) $$\ln\frac{P(t)}{P(0)}=\ln P(t)-\ln P(0)=\ln\frac{\sum p_t q_0}{\sum p_0 q_0}$$

[9] Divisia himself was already aware of this drawback and he proposed chain indices as a discrete approximation.
[10] Or: in which the individual price changes are weighted with constant base period weights.

such that we finally arrive at the Laspeyres index $P(t)/P(0) = \sum p_t q_0 / \sum p_0 q_0$. Of course P^L is path invariant as opposed to P^{Div}. It can be shown that P^P and P^F also might be regarded as special cases of the Divisia index.

d) Discrete time approximations and weights in Divisia's approach

Substituting forward differences $\Delta V_t = V_{t+1} - V_t = \sum p_{t+1} q_{t+1} - \sum p_t q_t$ for the differential dV (and correspondingly Δp_{it} and Δq_{it} for dp and dq) leads to

(2.5.17) $\Delta V_t = \sum_i q_{it} \Delta p_{it} + \sum_i p_{it} \Delta q_{it} + \sum_i \Delta p_{it} \Delta q_{it}$,

an equation equivalent to eq. 8 however with a mixed element $\sum_i \Delta p_{it} \Delta q_{it}$. It is reasonable therefore to define P_t and Q_t in such a way that

(2.5.18) $\frac{\Delta P_t}{P_t} = \frac{\sum q_t \Delta p_t}{\sum q_t p_t}$ and $\frac{\Delta Q_t}{Q_t} = \frac{\sum p_t \Delta q_t}{\sum q_t p_t}$. Using eq. 17 this gives

(2.5.19) $\frac{\Delta V_t}{V_t} = \frac{\Delta V_t}{\sum q_t p_t} = \frac{\Delta P_t}{P_t} + \frac{\Delta Q_t}{Q_t} + \frac{\sum \Delta p_t \Delta q_t}{\sum q_t p_t}$, or

(2.5.19a) $\frac{\Delta V_t}{V_t} = \left(\frac{P_{t+1}}{P_t} - 1 \right) + \left(\frac{Q_{t+1}}{Q_t} - 1 \right) + R_t$,

The residual term $R_t = \frac{\sum \Delta p_t \Delta q_t}{\sum q_t p_t}$ will tend to zero and can be neglected. Thus

(2.5.20) $\frac{P_{t+1}}{P_t} = \frac{\sum q_t p_{t+1}}{\sum q_t p_t} = P_{t+1}^{LC}$,

which is the Laspeyres link, and the corresponding *index* (comparing period t with 0) then is

(2.5.21) $\frac{P_{t+1}}{P_0} = \frac{P_1}{P_0} \frac{P_2}{P_1} \cdot \ldots \cdot \frac{P_{t+1}}{P_t} = P_1^{LC} P_2^{LC} \ldots P_{t+1}^{LC} = \overline{P}_{0,t+1}^{LC}$, and analogously

(2.5.22) $\frac{Q_{t+1}}{Q_t} = Q_{t+1}^{LC} = \frac{\sum q_{t+1} p_t}{\sum q_t p_t}$ and (7.2.5a) $\frac{Q_{t+1}}{Q_0} = \overline{Q}_{0,t+1}^{LC}$.

In a similar manner we may derive Paasche chain indices $\overline{P}_{0t}^{PC}$ (and $\overline{Q}_{0t}^{PC}$) by using *backward* differences $\Delta^* P_t = P_t - P_{t-1}$ and $\Delta^* p_{it} = p_{it} - p_{i,t-1}$ respectively

(2.5.23) $\frac{\Delta^* P_t}{P_t} = 1 - \frac{\sum q_t p_{t-1}}{\sum q_t p_t} = 1 - \frac{P_{t-1}}{P_t} = 1 - \left(\frac{P_t}{P_{t-1}} \right)^{-1} = 1 - \left(P_t^{PC} \right)^{-1}$.

This consideration does not, however, support the often heard statement, that the correctness of the chain approach has been proved by Divisia's formula:

> ".... the smaller we make the unit of time or space within which production or consumption takes place, the less actual production or consumption there will be to observe, and comparisons between these tiny units will become meaningless". (Diewert and Nakamura 1993, p. 3).

> "The problem with this approach is that economic data are almost never available as continuous time variables ... Hence for empirical purpose it is necessary to approximate the continuous time

Divisia price and quantity indexes by discrete time data. Since there are many ways of performing these approximations, the Divisia approach does not seem to lead to a definite result". (p. 23)[11].

2.6. Additive models, Stuvel's and Banerjee's index formulas

This is an additive approach to index numbers (fig. 2.6.1). The additive analysis is even on the microlevel not uniquely determined. Hence the following *two* equations with relatives $p_i^* = p_{it} / p_{i0}$ and $q_i^* = q_{it} / q_{i0}$ will hold

(*a1*) $v_{it} - v_{i0} = (v_{it} - v_{i0} p_i^*) + (v_{i0} p_i^* - v_{i0}) = v_{i0} p_i^* (q_i^* - 1) + v_{i0}(p_i^* - 1)$

(*a2*) $v_{it} - v_{i0} = (v_{it} - v_{i0} q_i^*) + (v_{i0} q_i^* - v_{i0}) = v_{i0} q_i^* (p_i^* - 1) + v_{i0}(q_i^* - 1)$.

Summation over all n commodities and division by $V_0 = \Sigma v_{i0} = \Sigma p_0 q_0$ yields (omitting the subscripts 0 and t for convenience of presentation) $V = V_{0t}$

(*A1**) $V - 1 = (V - P^L) + (P^L - 1) = P^L (Q^P - 1) + (P^L - 1)$

(*A2**) $V - 1 = (V - Q^L) + (Q^L - 1) = Q^L (P^P - 1) + (Q^L - 1)$.

Note that all equations presented so far are nothing but simple identities.

Figure 2.6.1: Stuvel's approach

a) Types of analysis

	microlevel	macrolevel
multiplicative approach	(*m*) $\frac{v_{it}}{v_{i0}} = \frac{p_{it}}{p_{i0}} \frac{q_{it}}{q_{i0}}$	(*M*) $V_{0t} = \frac{\sum v_{it}}{\sum v_{i0}} = P_{0t} Q_{0t}$
additive approach	(*a*) $\Delta v_i = v_{it} - v_{i0} = = A_i + B_i$	(*A*) $\Delta V = \sum v_{it} + \sum v_{i0} = = \sum A_i + \sum B_i = A + B$

terms A_i and B_i, (or A and B on the macrolevel) are supposed to measure price and quantity component

	comments
multiplicative approach	Decomposition is uniquely determined in the single commodity case (microlevel) by equation (*m*), but on the macrolevel (in eq. *M*) only V is determined uniquely, P and Q are not. The values of P and Q will depend on what indices are chosen for their measurement.
additive approach	The additive analysis is even for single commodities (microlevel) not uniquely determined. Two eq. possible (*a1* and *a2*) to measure price and quantity component

The two decompositions (equations) in additive analysis

microlevel	macrolevel
(*a1*) $\Delta v_i = v_{i0} p_i^* (q_i^* - 1) + v_{i0}(p_i^* - 1)$	(*A1*) $\Delta V = V_0 P(Q - 1) + V_0(P - 1)$
(*a2*) $\Delta v_i = v_{i0} q_i^* (p_i^* - 1) + v_{i0}(q_i^* - 1)$	(*A2*) $\Delta V = V_0 Q(P - 1) + V_0(Q - 1)$

p_i^* and q_i^* denote the price relative and quantity relative respectively.

[11] More important still, since the approximations "can differ *considerably* (in amount), the Divisia approach does not lead to a practical resolution of the price measurement problem" (p. 43). Emphasis and text in brackets added.

To define the two components of the (absolute) value change, A and B a decision has to be made as to which equation should be used, (*A1**) or (*A2**). Since there is no indication why one of the two equations should be preferred to the other Stuvel calculated the average of both equations and arrived at two equations

(*) $A + B = V_0(PQ - 1)$ and

(**) $A - B = V_0(Q - P)$

Furthermore since $V_t - V_0 = A + B$ (because of eq. A) we get $V_{0t} = (A+B)/V_0 + 1 = PQ$ which simply states that Stuvel's pair of index numbers will meet the factor reversal test. Substituting P by V_{0t}/Q we get $A - B = V_0\left(Q - \frac{V_{0t}}{Q}\right) = V_0 Q - \frac{V_t}{Q}$ and after some algebra we get the quadratic equation $Q^2 + \frac{B-A}{V_0}Q - V_{0t} = 0$,

and in a similar manner upon inserting V_{0t}/P for Q the following quadratic equation[12] in P

(2.6.1) $$P^2 + \frac{A-B}{V_0}P - V_{0t} = 0.$$

One of the two roots of this equation (taking into account (A-B)/$V_0 = Q^L - Q^P$ is given by

(2.6.2 $$P = \frac{B-A}{2V_0} + \sqrt{\left(\frac{B-A}{2V_0}\right)^2 + V_{0t}} = \frac{P_{0t}^L - Q_{0t}^L}{2} + \sqrt{\left(\frac{P_{0t}^L - Q_{0t}^L}{2}\right)^2 + V_{0t}} = P_{0t}^{ST}.$$

(2.6.3) $$Q = \frac{A-B}{2V_0} + \sqrt{\left(\frac{A-B}{2V_0}\right)^2 + V_{0t}} = \frac{Q_{0t}^L - P_{0t}^L}{2} + \sqrt{\left(\frac{Q_{0t}^L - P_{0t}^L}{2}\right)^2 + V_{0t}} = Q_{0t}^{ST}.$$

Properties of Stuvel's index formulas P_{0t}^{ST} and Q_{0t}^{ST}

axioms, "tests" and other criteria	
satisfied	violated; further remarks
all fundamental criteria, like dimensionality, commensurability, (strict) monotonicity, proportionality, factor reversal, time reversal test [2], consistency in aggregation and equality test satisfied	**linear homogeneity** [1] **not met**, no interpretation as means of price relatives and in terms of costs of a budget [3]

1) this is true also for the generalized Stuvel index except in case of P_{0t}^L, Q_{0t}^P and P_{0t}^P, Q_{0t}^L.
2) that means: Stuvels indices are "ideal" index functions.
3) the Stuvel indices are difficult to interpret economically.

It is easy to see why linear homogeneity is violated

$$P_{0t}^{ST}(\lambda) = \frac{\lambda P_{0t}^L - Q_{0t}^L}{2} + \sqrt{\left(\frac{\lambda P_{0t}^L - Q_{0t}^L}{2}\right)^2 + \lambda V_{0t}} \neq \lambda P_{0t}^{ST} \text{ because } \frac{\lambda P_{0t}^L - Q_{0t}^L}{2} \neq \lambda \frac{P_{0t}^L - Q_{0t}^L}{2}.$$

[12] the non-negative solution of which renders Stuvel's price index.

Alternative ways of deriving Stuvel's indices and a generalization of Stuvel's formulas

(*A1***)	$w(V - 1) = w(V - P^L) + w(P^L - 1)$
(*A2***)	$(1-w)(V - 1) = (1-w)(V - Q^L) + (1-w)(Q^L - 1)$

Special case P^{ST}/Q^{ST} is simply w = ½.

This idea is giving rise to a generalization of Stuvel's indices

Figure 2.6.2: An alternative way of deriving and interpreting P_{0t}^{ST} and Q_{0t}^{ST}

first condition: Find a pair of indices P (price index) and Q (quantity index) such that they pass factor reversal test $V_{0t} = P_{0t}Q_{0t}$ (equation2.6.5) **and**

second condition (**special**)	**second** condition (**general**)
give both types of additive decomposition (*A1*, *A2*) the same weight ½,	give additive decomposition (*A1*, *A2*) weights w and 1-w respectively
or equivalently: $P - P_{0t}^L = Q - Q_{0t}^L$ P should be equally away from P_{0t}^L as Q is away from Q_{0t}^L	equivalently: $w(P - P_{0t}^L) = (1-w)(Q - Q_{0t}^L)$ $(0 \le w \le 1)$ P_{0t}^{ST} and Q_{0t}^{ST} is the special case of w = ½ → **Generalized Stuvel** indices

(2.6.8) $$P_{0t}^{ST}(w) = \frac{P_{0t}^L - \frac{1-w}{w}Q_{0t}^L}{2} + \sqrt{\left(\frac{P_{0t}^L - \frac{1-w}{w}Q_{0t}^L}{2}\right)^2 + \frac{1-w}{w}V_{0t}}$$, and

(2.6.9) $$Q_{0t}^{ST}(w) = \frac{Q_{0t}^L - \frac{w}{1-w}P_{0t}^L}{2} + \sqrt{\left(\frac{Q_{0t}^L - \frac{w}{1-w}P_{0t}^L}{2}\right)^2 + \frac{w}{1-w}V_{0t}}$$.

Banerjee's factorial approach in index theory

(2.6.10a) $P_{0t} = \frac{P_t^*}{P_0^*}$ and $Q_{0t} = \frac{Q_t^*}{Q_0^*}$. The approach leads to the following system of equations (notation)

	price 0	price t
quantity 0	$Y_{00} = \sum p_0 q_0 = P_0^* Q_0^*$	$Y_{t0} = \sum p_t q_0 = P_t^* Q_0^*$
quantity t	$Y_{0t} = \sum p_0 q_t = P_0^* Q_t^*$	$Y_{tt} = \sum p_t q_t = P_t^* Q_t^*$

Figure 2.6.4: Banerjee's factorial approach (The system of equations)

	equation	derived from condition for
(a)	$(P_{0t}+1)(Q_{0t}+1)=V_{0t}+P_{0t}^{L}+Q_{0t}^{L}+1$	grand mean (μ)
(b)	$(P_{0t}-1)(Q_{0t}+1)=V_{0t}+P_{0t}^{L}-Q_{0t}^{L}-1$	factor price (α)
(c)	$(P_{0t}+1)(Q_{0t}-1)=V_{0t}-P_{0t}^{L}+Q_{0t}^{L}-1$	factor quantity (β)
(d)	$(P_{0t}-1)(Q_{0t}-1)=V_{0t}-P_{0t}^{L}-Q_{0t}^{L}+1$	interaction (γ)

Solution of the equations

Combination of equations	Price index = P_t^* / P_0^*	Quantity index = Q_t^* / Q_0^*
a and b	index of Marshall Edgeworth	index of Laspeyres Q_{0t}^{L}
a and c	index of Laspeyres P_{0t}^{L}	index of Marshall Edgeworth
a and d	no meaningful result	
b and c	index of Stuvel P_{0t}^{ST}	index of Stuvel Q_{0t}^{ST}
b and d	index of Laspeyres P_{0t}^{L}	**new formula** $Q_{0t}^{BA1}=Q_{0t}^{L}\frac{P_{0t}^{P}-1}{P_{0t}^{L}-1}$
c and d	**new formula** $P_{0t}^{BA1}=P_{0t}^{L}\frac{Q_{0t}^{P}-1}{Q_{0t}^{L}-1}$	index of Laspeyres Q_{0t}^{L}

Hence formulas of Stuvel, Laspeyres and Marshall-Edgeworth, as well as two new formulas appear as special cases of the factorial approach.

Another pair of indices derived by Banerjee from his "economic Theory" approach

index of	figure 2.6.4	economic theory
prices	$P_{0t}^{BA1}=P_{0t}^{L}\frac{Q_{0t}^{P}-1}{Q_{0t}^{L}-1}$	$P_{0t}^{BA2}=\frac{P_{0t}^{P}\left(P_{0t}^{L}+1\right)}{P_{0t}^{P}+1}$
quantities	$Q_{0t}^{BA1}=Q_{0t}^{L}\frac{P_{0t}^{P}-1}{P_{0t}^{L}-1}$	$Q_{0t}^{BA2}=\frac{Q_{0t}^{P}\left(Q_{0t}^{L}+1\right)}{Q_{0t}^{P}+1}$

It can be shown that both Banerjee-2 indices are bounded by the respective Laspeyres- and Paasche-indices P^L, Q^L and P^P, Q^P. Given that $P_{0t}^{L}=(P_{0t}^{P}+\Delta)>P_{0t}^{P}, \Delta>0$ we obtain $\left(P_{0t}^{P}+1\right)P_{0t}^{BA2}=P_{0t}^{P}\left(P_{0t}^{P}+1+\Delta\right)$ and thus $P_{0t}^{BA2}=P_{0t}^{P}+\frac{\Delta P_{0t}^{P}}{P_{0t}^{P}+1}>P_{0t}^{P}$ and in just the same manner we get $P_{0t}^{BA2}<P_{0t}^{L}$ in this situation ($P_{0t}^{P}<P_{0t}^{L}$), and $P_{0t}^{L}<P_{0t}^{BA2}<P_{0t}^{P}$ in the case $P_{0t}^{L}<P_{0t}^{P}$.

The Banerjee-1 indices are not bounded by the respective Laspeyres- and Paasche-indices, and these indices can even yield completely absurd results: whenever a Paasche quantity index displays decreasing quantities ($Q^P < 1$) and the Laspeyres quantity index increasing quantities ($Q^L > 1$) then P^{BA1} will be negative (!). This applies also mutatis mutandis to P^{BA1} with reference to P^P and P^L.