Bortkiewicz on Chain Indices and Irving Fisher's Reversal Tests, A Historical Note and a Disapproval of the Time Reversal Test

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Ladislaus von Bortkiewicz has criticized chain indices and Irving Fisher's time and factor reversal tests. However, his arguments (published in German articles) though well ahead of his time and still relevant today are widely fallen in oblivion. He was not the only German statistician who criticized Fisher' approach but the first who extensively and successfully used mathematics to substantiate it and to derive a formula for the "chain drift". We present his ideas together with some own arguments against the time reversal test. To study Bortkiewicz's and other criticisms of Fisher's reversal tests and chain indices remains worthwhile, because reference to the time reversal test is still unshakably popular today and it only recently became mandatory to compile chain indices in official statistics.

* In this section we no longer only report LvB's views but add some of our own considerations regarding the TR

1. Chain indices

a) Bortkiewicz as "non-chainer"

Ladislaus von Bortkiewicz (1868 – 1931, LvB for short) argued against chain indices (or "the chain system" ["Kettensystem"] as he used to call it) on the following grounds:

1. Chain indices (CI) ***violate*** (the axiom of) ***proportionality*** (and thereby also identity).
2. With CI the ***base period*** of an index, in general kept constant for a couple of periods, is totally ***deprived of its relevance***, and as there is no limit for the length of a chain this "*endows the chain system with eternal life"* (as LvB has put it a bit ironically):
   a) though on the one hand we may be happy to get rid of the trouble with choosing from time to time a suitable new base year (as it ought to be a "normal" year),[1]
   b) on the other hand, however, this is far from advantageous, because
      - it is just the constant weight "base" that makes subsequent index numbers comparable among themselves, so that with chain indices there is no longer a constant measuring rod that guarantees that like is compared with like, and
      - the continual (in each period) updating of weights or "infusion of new weights" "adds so to say a foreign element to the compilation of the price index"[2] which makes the chain index path dependent.[3]
3. ***Transitivity*** or circularity (or "intercalation" as it was used to be called in his days) requires $P_{0t} = P_{0k}P_{kt}$ for *any* period k (not only a specific intermediate period), and LvB clearly realized that chain indices not only violate proportionality but also transitivity,

[1] This has, however, to be set against the need to define new empirical weights every new period in CIs.
[2] "*wird gleichsam ein fremdes Element in die Ermittlung der gesuchten Preisindexzahl hineingetragen*" – we henceforth quote German texts preferably in italics – (B 1923/II, 217). By this (a "foreign element" {*fremdes Element*} is introduced in the index compilation") LvB meant prices p_{ik} and in particular also quantities q_{ik}, i = 1, …,n in the *intermediate [or "intercalated"] period k* in an attempt to measure the change of prices (not quantities) between *0 and t* (0 < k < t). "Chaining" P_{0k} an P_{kt} to get a P_{0t} as product P_{0k} P_{kt} creates path dependence.
[3] LvB did not yet use the term "path dependence, but he was fully aware of the phenomenon as such.

because the typical indirect comparison between 0 and t via k of $\overline{P}_{0t} = P_{0k}P_{kt}$, the chain index[4] will in general differ from the direct[5] comparison P_{0t} so that $D^P_{0t} = \overline{P}_{0t}/P_{0t} \neq 1$ (the "chain drift" of a price index P). In other words, comparisons over time (temporal aggregations) become inconsistent, or – what indeed may sound a bit strange – a CI is gained by chaining (multiplying), but is not "chainable" [transitive]). For LvB

a) an indirect comparison $P_{0k}P_{kt}$ (via k) not only differs from the direct comparison, resulting in chain drift
b) it may well also *differ from other equally justifiable indirect comparisons* via r for example so that $P_{0k}P_{kt} \neq P_{0r}P_{rt}$,[6] and
c) as soon will be shown below, he had put some effort in exploring mathematically the cause and nature of chain drift (its amount, and whether and under which conditions it will increase or decrease with the passage of time).

Conspicuously there is always *only one direct index* P_{0t}, but as a rule *number of* possible *indirect (chained) indices* $\overline{P}_{0t}$. Inconsistencies above, $P_{0k}P_{kt} = \overline{P}_{0t} \neq P_{0t}$ and $P_{0k}P_{kt} \neq P_{0r}P_{rt}$ are only two obvious manifestations of the inherent ambiguity of indirect comparisons known as "path dependence". There are other less obvious ones.[7]

It is noteworthy that according to B 1924/II, 219 and B 1927), 749 it was the German J. Lehr (and then A. Marshall) who first proposed chain indices (see also v.d.Lippe 2013, 357) and that LvB not only disapproved chain indices because of their failing "pure price comparison", but rather by a number of "formal" (axiomic) shortcomings.[8] By contrast the dominant style of approach to index numbers at that time in Germany was more of the verbal or "philosophical" sort (preferably in the case of the numerous less renowned economists and statisticians).

b) Proportionality

That chain indices violate this axiom which implies $P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t) = P(\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_0,\mathbf{q}_t) = \lambda$, was already observed by Fisher himself as rightly acknowledged by LvB (B 1924/II, 217). However, Fisher and LvB vehemently disagreed in the emphasis they placed on the importance of this axiom as opposed to the reversal tests. In LvB's view to fail proportionality is an enormous disadvantage of an index, as it entails also no "multi-period identity".[9] Fisher in turn set great store by his "Great Reversal Tests" (as he himself has put it), while LvB on the other

[4] In order to keep a chain index distinct from a direct index, the chain index is denoted with a bar on top of P.
[5] LvB explicitly made the distinction (as also the present author does) between "indirect [chained]" indices (in our notation with bar) and the corresponding "direct" indices (without bar) B 1924/II, 214. He did not use those misleading terms like "fixed base" or fixed weighted" which are in use nowadays to denote a "direct" index.
[6] This is simply another manifestation of intransitivity. LvB demonstrated this with an example (see ex.2) of international comparisons showing that it matters which country (k or r) is used as a country via which two countries (here 0 and t) are indirectly compared (or "linked"). As is well known, without truly transitive index functions consistency can only be achieved by either comparing any two countries via an artificial third country (or "bloc country") or by averaging over all possible "third" countries.
[7] LvB possibly not yet saw a third type of inconsistency, viz. that the result of chaining also depends on how the interval (0, t) is partitioned into subintervals. So for example the product (chain) $P_{01}P_{12}\ldots P_{56}$ (annual linking) will in general differ from $P_{02}P_{24}P_{46}$ (biannual linking). As mentioned above (the "foreign element") LvB characterized "path dependence" already quite clearly, saying that the result necessarily depends on what happens with prices and quantities in intermediate periods: It turns out "*überhaupt als mißlich*" (= on the whole as uncomfortable) that the result "*davon abhängig gemacht wird, wie sich die Preisverhältnisse und gegebenenfalls auch die Mengenverhältnisse in den dazwischen liegenden Zeiträumen gestaltet haben*" (= is made dependant on what happened with prices and maybe also quantities in the interval between the two points in time) (B 1927, 749).
[8] Another prominent German speaking author who deserves being mentioned here was Gottfried Haberler (1900 - 1995). He expressed in particular distrust in Fisher's predilection for the FR (Haberler 1927), struggling much with the notion of a "collective" (national vs. household's) purchasing power. He later (1936) moved to the USA.
[9] LvB made this point not only in part I of his famous paper, but again also in part III (B 1924/III, 510). And this clearly was a bone of contention with Irving Fisher who had an equally strong preference for his reversal tests.

hand viewed those tests with no small suspicion. LvB demonstrated violation of proportionality by chain indices with a numerical example which goes as follows:

> **Ex1:** Assume three price and quantity row-vectors respectively (for three periods 0,1,2), each for two goods as follows $\mathbf{p}_0' = [1 \quad 2]$, $\mathbf{p}_1' = [2 \quad 1]$, $\mathbf{p}_2' = [3 \quad 6]$, and $\mathbf{q}_0' = [10 \quad 16]$, $\mathbf{q}_1'$ $= [30 \quad 18]$, $\mathbf{q}_2' = [20 \quad 4]$. Note that both price relatives amount to $p_{i2}/p_{i0} = \lambda = 3$ (i = 1, 2) so that it a price index should yield $P_{02} = \lambda = 3$ as well. While all three direct indices, Laspeyres (L), Paasche (P) and Fisher (F) pass this "proportionality test", because $P_{02}^L = P_{02}^P$ $= P_{02}^F = \lambda = 3$, none of the three chain indices does: $\overline{P}_{02}^L = 2.175$, $\overline{P}_{02}^P = 2.256$, $\overline{P}_{02}^F = 2.215$.

c) Transitivity

In contrast to proportionality as regards transitivity LvB's position was not very explicitly pro or con. He viewed transitivity slightly beneficial because it frees us from the annoying choice of an appropriate "base period" (B 1924/II, 216). On the other hand, however, as there are only very few truly transitive index formulas he clearly saw that transitivity is a very demanding if not *unduly* restrictive property of an index function which cannot be achieved unless other more important properties are sacrificed so that on the one hand

- transitivity is likely to rule out many otherwise quite useful index functions (such as P_{0t}^L, P_{0t}^P, or even P_{0t}^F), and on the other hand it
- could make us choose unfavorable index formulas (his point in case here was the unit value index P^{DR} of the "philosopher" Drobisch (more about P^{DR} soon below).[10]

LvB's stance on transitivity appeared not sufficiently pronounced and worked out in detail. It seems that index theory at that time had not yet a clear idea of the relationship between time reversibility (TR) and transitivity (circularity). We now know that they are related as follows:

if the circular test and identity is satisfied $\longrightarrow$ then also the time reversal test

But the converse is not true (thus TR is less demanding than transitivity), and the best known example for this is Fisher's "ideal index" P^F which passes the TR but not the circular-test.[11] This raises the questions

1. whether there is some extra-benefit of the much more demanding transitivity, and
2. why LvB saw an advantage in transitivity (because of the independence of the base) and yet argued at the same time quite vigorously against TR.

Ad 1: It is well known that Irving Fisher dropped circularity (after having realized that his index fails it) but continued to consider TR as prerequisite. There is no contradiction as he simply might not have seen enough extra-benefit to justify the highly restrictive transitivity.

Ad 2: sympathy for transitivity because of irrelevance of the base and criticism of TR is also not contradictory when there are de-merits of TR not shared by transitivity.

A note on weak[12] identity: This requires not only $\mathbf{p}_t = \mathbf{p}_0$, but $\mathbf{p}_t = \mathbf{p}_0$ *and* $\mathbf{q}_t = \mathbf{q}_0$. An index satisfying transitivity, but weak identity only is the unit-value index of Drobisch $P_{0t}^{DR} = \tilde{\mathbf{p}}_t / \tilde{\mathbf{p}}_0$

[10] Apparently for LvB Drobisch (a math-professor!) was not on par with him and economists and statisticians like Edgeworth, Laspeyres etc. There also was quite a bit of altercation between Drobisch and Laspeyres (unjustified attacked by Drobisch). For more about P^{DR} and the early index theory in Germany see v.d.Lippe 2013.

[11] A necessary and sufficient condition for transitivity of an index to hold is that the index can be expressed as a ratio ($\mathbf{p}_t$)/f($\mathbf{p}_0$), and P^F cannot be written as f($\mathbf{p}_t$)/f($\mathbf{p}_0$).

[12] A "weak" property always differs from the corresponding "strict" one in that it requires additional assumptions. When a *general* condition is failed the more *special* situation of "weak" the condition can yet be met.

with the unit value $\tilde{\mathbf{p}}_t = \Sigma p_{it}q_{it}/\Sigma q_{it}$ and $\tilde{\mathbf{p}}_0$ defined analogously.[13] By the way, in LvB's view P^{DR} serves as a strong argument against the factor reversal test (FR) because the "factor antithesis" (as Fisher put it) of P_{0t}^{DR} is the Dutot quantity index $Q_{0t}^{D} = \Sigma q_t/\Sigma q_0$ (as $P^{DR}Q^{D} = V_{0t} = \Sigma p_t q_t/\Sigma p_0 q_0$), and that both indices, P^{DR} and Q^{D}, were for him of limited if any value at all.

Finally LvB also ridiculed a bit Fisher's well known inconsistent thoughts on transitivity:

- On the one hand realizing that his "ideal index" is not transitive Fisher turned from approval to vehement disapproval of transitivity when he even said that to (exactly) fulfill the "circular test" is almost a proof that the formula in question is "erroneous",
- on the other hand, however, Fisher considered an index formula preferable to the extent to which it *approximately* meets transitivity (the closer you come, the better).

Such ambiguity is owed to Fisher's peculiar notion of *bias* B (see below p. 13), which he applied, however, primarily to the deviation of index functions (P and/or Q) from TR or FR in the sense of $P_{0t}P_{t0} = B - 1$ and $P_{0t}Q_{0t} = B - V_{0t}$, B 1924/II, 212f.[14]

To sum up: while LvB's evaluation of transitivity – based primarily on the advantage of independence of the base period – does not seem to be sufficiently clear, it is (as mentioned) quite an accomplishment of LvB to have shown that chaining does not yield transitivity.

d) The amount of "chain drift"

LvB studied chain indices $\overline{P}_{0t}^{L}$. $\overline{P}_{0t}^{P}$, and $\overline{P}_{0t}^{F}$ referring to the same data, Fisher already had used for demonstration purposes (B 1924/II, 214) and he was – unlike Fisher[15] – able to develop a formula for the chain-drift. To this end he made use of his famous covariance formula to compare two linear index functions.[16] Contrary to what is often written he already presented his formula *in its general form*,[17] not only in the well known special form which explains the positive/negative difference between a (direct) Paasche price index (P^P) and a Laspeyres price index (P^L) with reference to the weighted (weights $p_0q_0/\Sigma p_0q_0$) covariance C between price and quantity relatives $x_{i,0t} = p_{it}/p_{i0}$ and $y_{i0t} = q_{it}/q_{i0}$ ($P_{0t}^{P} < P_{0t}^{L}$)[18]

(1) $$C = \mathrm{Cov}(x_{0t}, y_{0t}) = V_{0t} - P_{0t}^{L}Q_{0t}^{L},$$

where V_{0t} denotes the value ratio $\Sigma p_t q_t/\Sigma p_0 q_0$ also given by $V_{0t} = P_{0t}^{L}Q_{0t}^{P} = P_{0t}^{P}Q_{0t}^{L}$ and

(1a) $$C = \sum\left(\frac{p_{it}}{p_{i0}} - P_{0t}^{L}\right)\left(\frac{q_{it}}{q_{i0}} - Q_{0t}^{L}\right)\frac{p_{i0}q_{i0}}{\sum p_{i0}q_{i0}} = Q_{0t}^{L}\left(P_{0t}^{P} - P_{0t}^{L}\right),$$ so that

(2) $$P_{0t}^{P} - P_{0t}^{L} = \frac{C}{Q_{0t}^{L}}.$$

By contrast to LvB's *weighted* covariance Fisher made use of an *unweighted* covariance

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[13] LvB discarded the P^{DR} index mainly because it fails *strict* identity (see B 1924/III, 510. There he even said that P^{DR} does not deserve to be called "price index").

[14] Interestingly LvB also examined "crossing" (in the sense of Fisher) of formulas, and he found – contrary to Fisher's conjecture and to Fisher's plea for crossing of indices – that this operation does not guarantee less deviation from transitivity (i.e. a smaller drift), see B 1924/II, 216.

[15] According to LvB Fisher tried to find a relation between Laspeyres and Paasche using an unweighted covariance, whereas LvB made use of weights $p_0q_0/\Sigma p_0q_0$ (as will be seen in eq. 1a).

[16] See Annex 1 of this paper for the original version of LvB's formula in LvB's notation and our notation. An index is said to be linear when it can be written as ratio of scalar products of a price- and a quantity vector.

[17] Until recently when I had not yet read the original German paper of Bortkiewicz I used to follow other authors, who (erroneously, as it now turns out) believe that LvB only compared the Paasche and Laspeyres price index respectively with his covariance formula (eqs. 1, 1a and 2). However, this is only a special variant of his theorem he already developed in its "generalized" form in B 1923/I, 374 – 376 (also in v.d.Lippe 2007, 194f.)

[18] In order to simplify the presentation we will also refer to p_t/p_0 and q_t/q_0 only and drop the subscript i.

$\sum \frac{1}{n}\left(\frac{p_{it}}{p_{i0}} - P^C_{0t}\right)\left(\frac{q_{it}}{q_{i0}} - Q^C_{0t}\right) = \frac{1}{n}\sum \frac{p_{it}q_{it}}{p_{i0}q_{i0}} - \left(2 - \frac{1}{n}\right)P^C_{0t}Q^C_{0t}$ with the Carli price index $P^C_{0t} = \frac{1}{n}\sum \frac{p_{it}}{p_{i0}}$ and Q^C_{0t} the Carli quantity index defined correspondingly.[19]

LvB used his general relation (of which eq. 1a is a special case only) in order to derive the drift[20] $D^{PL}_{02} = \overline{P}^L_{02}/P^L_{02}$. Here the decisive element is the covariance between the price growth factor $x_{i,12} = p_{i2}/p_{i1}$ and the *cumulated* quantity change $y_{i,01} = q_{i1}/q_{i0}$.

Using $\bar{x}_{12} = \sum \frac{p_2}{p_1}\frac{p_1q_0}{\sum p_1q_0} = \frac{\sum p_2q_0}{\sum p_1q_0} = P^L_{02} / P^L_{01} = P_{12(0)}$ and $\bar{y}_{01} = \sum \frac{q_1}{q_0}\frac{p_1q_0}{\sum p_1q_0} = Q^P_{01}$ we arrived in v.d.Lippe 2001, 141 – 145, and v.d.Lippe 2007, 480 – 484 at

(3) $\quad \mathrm{Cov}(x_{12}, y_{01}) = \sum (x_{i,12} - \bar{x}_{12})(y_{i,01} - \bar{y}_{01})\frac{p_1q_0}{\sum p_1q_0} = \frac{\sum p_2q_1}{\sum p_1q_0} - \bar{x}_{12}\bar{y}_{01} = \frac{Q^P_{01}}{P^L_{01}}\left(P^L_{01}P^L_{12} - P^L_{02}\right)$

$= Q^P_{01}\left(P^L_{12} - P^L_{12(0)}\right)$. Note that 3) is in fact equivalent to $\mathrm{Cov}(x_{12}, y_{01})\frac{P^L_{01}}{Q^P_{01}} = P^L_{01}P^L_{12} - P^L_{02} = \overline{P}^L_{02} - P^L_{02}$ (in our notation)[21]. Thus $D^{PL}_{02} = \frac{\overline{P}^L_{02}}{P^L_{02}} = \frac{\mathrm{Cov}(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1$ is the "chain drift" of $\overline{P}^L_{02}$.

This may easily be generalized as follows to get chain drifts of $\overline{P}^L_{03}$, $\overline{P}^L_{04}$ etc.[22]

growth factors (t-1 $\rightarrow$ t) of ***prices (x)***	***relatives*** (cumulative changes 0 $\rightarrow$ t) of ***quantities (y)***
$x_{i,12} = \frac{p_{i2}}{p_{i1}}$, $x_{i,23} = \frac{p_{i3}}{p_{i2}}$, $x_{i,34} = \frac{p_{i4}}{p_{i3}}$,… etc.	$y_{i,01} = \frac{q_{i1}}{q_{i0}}$, $y_{i,02} = \frac{q_{i2}}{q_{i0}}$, $y_{i,03} = \frac{q_{i3}}{q_{i0}}$,... etc.

Thus the chain drift depends on a *weighted* covariance between *growth factors of prices* $x_{t-1,t}$ and r*elatives* (i.e. *cumulated* changes) *of quantities* $y_{0,t-1}$ as follows (omitting i for convenience of presentation).

In a similar manner we get x_{23} and y_{02} defined accordingly using weights $p_2q_2/\Sigma\, p_2q_2$

(4) $\quad \mathrm{Cov}(x_{23}, y_{02}) = \sum (x_{i,23} - \bar{x}_{23})(y_{i,02} - \bar{y}_{02})\frac{p_2q_0}{\sum p_2q_0} = \frac{Q^P_{02}}{P^L_{02}}\left(P^L_{02}P^L_{23} - P^L_{03}\right) = Q^P_{02}\left(P^L_{23} - P^L_{23(0)}\right)$,

where $\bar{x}_{23} = P^L_{03} / P^L_{02} = P_{23(0)}$, and $\bar{y}_{02} = Q^P_{02}$.

Note that $P^L_{02}P^L_{23} \neq \overline{P}^L_{03} = P^L_{01}P^L_{02}P^L_{23}$. Likewise

(5a) $\quad \mathrm{Cov}(x_{34}, y_{03}) = \sum (x_{i,34} - \bar{x}_{34})(y_{i,03} - \bar{y}_{03})\frac{p_3q_0}{\sum p_3q_0} = \frac{Q^P_{03}}{P^L_{03}}\left(P^L_{03}P^L_{34} - P^L_{04}\right) = Q^P_{03}\left(P^L_{34} - P^L_{34(0)}\right)$,

[19] See B 1923/I, 396. Fisher thereby failed showing why the difference between Laspeyres (P^L) and Paasche (P^P) should necessarily be small. LvB called in question this conjecture of a generally small difference $P^P - P^L$, and it was only he who could properly demonstrate, how the difference $P^P - P^L$ is determined by a covariance. Note, however, that also LvB thought that C will as a rule tend to be negative, so that we often should have $P^P < P^L$.

[20] Drift in this context usually refers to a relation between an indirect index (chain index) and its corresponding direct index. We present here only LvB's proof that P^L is not "chainable". He also derives a similar proof for P^P.

[21] Eq. 3 above is LvB's eq. 42 in B 1924/II, 211.

[22] v.d.Lippe 2007, 482f. LvB also made use of the formula in order to show (in a surprisingly simple way) that both indices P^P and P^L are not transitive B 1924/II, 211f.). He demonstrated this with empirical (German wholesale) price indices, and he also found out– contrary to Fisher's conjecture – that the "ideal index" of Fisher $P^F = (P^LP^P)^{1/2}$ does not necessarily provide a smaller deviation from transitivity (meaning that it not necessarily represents a better approximation to a truly transitive index) than either index (P^L and P^P) taken in isolation (B 1924/II, 212. A similar result also applies to "crossing" of formulas in general B 1924/II, 226).

where $\bar{x}_{34} = P^{L}_{34(0)} = P^{L}_{04}/P^{L}_{03}$, $\bar{y}_{03} = Q^{P}_{03}$ etc., and D^{PL}_{03}, D^{PL}_{04} etc. defined analogously to eq. 4: Again $P^{L}_{03}P^{L}_{34} \neq \bar{P}^{L}_{04}$ once $D^{PL}_{03} \neq 1$ and $D^{PL}_{04} \neq 1$.

Now the cumulative structure of the drift as a function of the (temporal) covariance is quite obvious and easily visible in table 1,[23] which shows that the drift itself develops like a chain where $D^{PL}_{03} = D^{PL}_{02}D^{PL}_{23}$, $D^{PL}_{04} = D^{PL}_{03}D^{PL}_{34}$ … will increase or decrease depending on the positive or negative sign of the covariance Cov( ) in $D^{PL}_{t-1,t} = \frac{\mathrm{Cov}(x_{t-1,t}, y_{0,t-1})}{\bar{x}_{t-1,t} \cdot \bar{y}_{0,t-1}}$ with weights $p_{t-1}q_0/\Sigma\, p_{t-1}q_0$.

Table 1: Drift function D^{PL} of $\bar{P}^{L}_{0t}$, the chained Laspeyres price index[24]

t	drift function D^{PL} and covariance cov(x, y)
t = 2, D^{PL}_{02}	$\frac{\mathrm{Cov}(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1 = \frac{P^{L}_{01}P^{L}_{12}}{P^{L}_{02}} = D^{PL}_{01}D^{PL}_{12}$ where $D^{PL}_{01} = 1$, and $D^{PL}_{12} = \frac{\mathrm{Cov}(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1$
t = 3, D^{PL}_{03}	$D^{PL}_{02}\left(\frac{\mathrm{Cov}(x_{23}, y_{02})}{\bar{x}_{23} \cdot \bar{y}_{02}} + 1\right) = D^{PL}_{02}\frac{P^{L}_{02}P^{L}_{23}}{P^{L}_{03}} = \left(\frac{\mathrm{Cov}(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1\right)\left(\frac{\mathrm{Cov}(x_{23}, y_{02})}{\bar{x}_{23} \cdot \bar{y}_{02}} + 1\right)$
t = 4, D^{PL}_{04}	$D^{PL}_{03}\left(\frac{\mathrm{Cov}(x_{34}, y_{03})}{\bar{x}_{34} \cdot \bar{y}_{03}} + 1\right) = \left(\frac{\mathrm{Cov}(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1\right)\left(\frac{\mathrm{Cov}(x_{23}, y_{02})}{\bar{x}_{23} \cdot \bar{y}_{02}} + 1\right)\left(\frac{\mathrm{Cov}(x_{34}, y_{03})}{\bar{x}_{34} \cdot \bar{y}_{03}} + 1\right)$

There is obviously an underlying *system* in table 1. The problem, however, with his table is to give a meaningful interpretation to the sequence of covariances because x refers to the change of *prices* between two *adjacent periods*, t-1 and t, while y is a measure of the *cumulative change of quantities* over an interval (0, 1,…, t-1). In the well known "usual" covariance C in (1) the variables refer to *the same interval in time*. Now, however, it is a lot more difficult to draw general conclusions concerning the sign and amount of the drift[25]. Already LvB took the view that this lack of a "theory" of the drift is a severe drawback of the chain system: Note

1. the relevant covariance may well change sign, such that for example $\bar{P}^{L}_{02} > P^{L}_{02}$ (or $D^{PL}_{02} > 1$) because of Cov(x_{12}, y_{01}) > 0 and at the same time $\bar{P}^{L}_{03} < P^{L}_{03}$, because Cov($x_{23}$, y_{02}) < 0;
2. in order to have no drift we need a sequence of identically zero-correlations (covariances) and/or constant prices/quantities ($x_{t-1,t} = y_{0,t-1}=1$) thoughout,[26] and
3. as it only later (not by LvB already) was found out, the drift depends on the time series of prices p_{it} (and quantities q_{it} respectively)[27] in which p_{it} (and q_{it}) may either be monotonical-

[23] This is table 7.2.5 in v.d.Lippe 2007, 483. However, LvB did not make clear (as we do above) that the drift D_{03}, D_{04} etc. of a chain index is again (like a chain index itself) a product of two, three etc. drift factors.

[24] D^{PL} denotes the drift of a Laspeyres price index. We refrain from discussing drifts of other *price* indices (D^{PP}, Paasche for example) but will shortly mention below D^{QL}, the drift of a Laspeyres *quantity* index.

[25] The same is true for the "spread" (or "gap") between the direct Laspeyres and Paasche index formula above.

[26] LvB explicitly drew attention to this problem (of interpreting the covariance), writing: "*Hieraus ersieht man, dass bei der in Frage stehenden Methode (von den Fällen abgesehen, in denen entweder Preisrelationen p_k/p_i oder die Mengenrelationen q_i/q_h nicht variieren….) der betreffende Korrelationskoeffizient gleich Null sein müsste, damit das Interkalationskriterium erfüllt sei. Bemerkenswert ist es hierbei, dass es auf die Korrelation nicht zwischen pi/ph und qi/qh oder p_k/p_i und q_k/q_i, sondern zwischen p_k/p_i und q_i/q_h ankommt*"(In LvB's notation periods h, i, and k correspond to our periods 0, 1, and 2) B 1924/II, 211. The "*in Frage stehenden Methode*" (method in question) here, in this quotation, is of course the chain index (the "Kettensystem").

[27] The problem with such a statement is, however, that empirical findings concerning the shape of time series of prices and quantities are based on *indices* of prices and quantities: given a cyclical movement a chain index might not reflect cycles but instead possibly rather a monotonically growing bias upward or downward.

ly increasing or decreasing, or oscillating around a (possibly zero) trend (in which case a chain index may well drift away further and further from the corresponding direct index).[28]

Thus it is difficult if not outright impossible – even with the interesting drift-formulas of LvB at hand – to infer some simple general conclusions as regards the development of a chain-drift from a visual inspection of the empirical time series of prices and quantities only. It may be interesting to note in passing that LvB also derived a formula for the drift of the Laspeyres *quantity* index $D^{QL}_{01} = \overline{Q}^{L}_{02}/Q^{L}_{02}$ by interchanging prices and quantities. It turns out that D^{QL}_{01} now can be explained using the covariance between $y_{i,12} = q_{i2}/q_{i1}$ and $x_{i,01} = p_{i1}/p_{i0}$

$$Cov(y_{12}, x_{01}) = \sum (y_{i,12} - \bar{y}_{12})(x_{i,01} - \bar{x}_{01}) \frac{q_1 p_0}{\sum q_1 p_0} = \frac{P^P_{01}}{Q^L_{01}}\left(Q^L_{01}Q^L_{12} - Q^L_{02}\right) = P^P_{01}\left(Q^L_{12} - Q^L_{12(0)}\right)$$

where $\bar{x}_{01} = P^P_{01}$, and $\bar{y}_{12} = Q^L_{02}/Q^L_{01}$, instead of $Cov(x_{12}, y_{01}) = Q^P_{01}\left(P^L_{12} - P^L_{12(0)}\right)$ determining the drift D^{PL}_{01} (B 1924/II, 211).

2. Time reversal test (TR)

Fisher's TR continues to play an important part in index theory as a test a good index function (ostensibly) ought to pass. It is widely used in order to justify a preference for Fisher's ideal index and TR also helped to rule out for example Carli's index (as "biased upwards") in the case of "low level aggregation" (unweighted indices). So a closer look at TR is worthwhile. We begin with arguments used by LvB and then add some of our own arguments against TR.

a) Bortkiewicz's criticism of the TR

LvB advanced mainly three arguments against the relevance and usefulness of the TR:

1. TR is a purely formal (or "mechanistic") test and it is easy to find a formula (however pointless it economically may be) that complies with this test, and
2. there is no reason why in TR in addition to "reversing" prices $\mathbf{p}_t \leftrightarrow \mathbf{p}_0$ also quantities should be reversed ($\mathbf{q}_t \leftrightarrow \mathbf{q}_0$) simultaneously, and
3. reversal tests are motivated by nothing but dubious analogies and intuitive appeal.

1. TR is a purely formal test

For Irving Fisher both tests, TR and FR were also "finders of formulae". Fisher called the index $P^{(T)}_{0t} = 1/P_{t0}$, "time antithesis" of P_{0t} and he considered the index $P^{(T)}_{0t}$, a byproduct of TR, a new index formula no less useful than P_{0t}.[29] It can easily be seen that P^P is the time antithesis of P^L since $1/P^L_{t0} = P^P_{0t}$ (just like P^L is the time antithesis of P^P). From this it follows[30] that we have two products of index formulas, A and B where $A = P_{0t}P^{(T)}_{0t} = P_{0t}/P_{t0}$ and $B = P_{t0}P^{(T)}_{t0} = P_{t0}/P_{0t}$ so that B is the time reversed term A. Hence $P^{(TR)}_{0t} = \sqrt{A}$ gives

(6) $\quad P^{(TR)}_{0t} = \sqrt{P_{0t}\dfrac{1}{P_{t0}}} = \sqrt{P_{0t}P^{(T)}_{0t}}$, which is a time reversible index on the basis of P_{0t} just like $P^{(TR)}_{t0} = \sqrt{B}$ is a time reversible variant of P_{t0} so that $P^{(TR)}_{0t}P^{(TR)}_{t0} = \sqrt{A \cdot B} = \sqrt{1} = 1$.

Note that $P^{(T)}$ should be kept distinct from $P^{(TR)}$: unlike $P^{(TR)}_{0t}$ the index $P^{(T)}_{0t}$ is in general not time reversible ($P^{(T)}_{0t}P^{(T)}_{t0} \neq 1$) unless the underlying index P_{0t} is itself time reversible. The

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[28] For this well known observation (or shortcoming of chain indices) see v.d.Lippe 2007, 464.

[29] P_{t0} is gained from P_{0t} by interchanging subscripts 0 an t in prices and quantities in the P_{0t} formula.

[30] I could not verify who (Fisher or LvB) first saw this. Anyway, I took the formulas from a paper of LvB.

message now is: for *any* index formula irrespective of its meaning, even for a quite nonsensical one, say $P_{0t}^{\#}$ we can get a corresponding time reversible index function with $\sqrt{P_{0t}^{\#}/P_{t0}^{\#}}$. So to pass the TR as such is not really a remarkable feat for an index.[31] The same is true for the factor reversal test (FR) where also (6) in Annex 2 *works with any formula* $P_{0t}^{\#}$. Just like (5) requires to take the geometric mean of an index and its "time antithesis" $\left(P_{t0}^{\#}\right)^{-1}$ we simply have to take the geometric mean of an index $P_{0t}^{\#}$ and its "factor antithesis" $V_{0t}/Q_{0t}^{\#}$ in order to get a factor reversible counterpart of $P_{0t}^{\#}$.

It can easily be seen that (5) applied to the price index of Laspeyres P^L and Paasche P^P respectively yields Fisher's "ideal" index $P^F = (P^L P^P)^{-1}$ since P^L is the time antithesis of P^P and vice versa. In this case the new index, that is P^F, is a sensible and meaningful index in its own right. But this need not be the case, and exactly here the critique of LvB and others comes in: Fisher's approach is purely mechanistic or "formal" because it is always possible – for any price index formula $P_{0t}^{\#}$ whatsoever, whether economically meaningful or not – to find a price index which is time reversible, factor reversible or both, time *and* factor reversible.[32]

2. Time reversal (TR) in two steps

In LvB's view Irving Fisher has not given sufficient and convincing arguments in favour of his TR (B 1923/I, 394). In particular he did not justify why interchanging prices should also automatically (and simultaneously) entail interchanging quantities as well. In contrast to Fisher LvB suggested a two-steps-procedure for what he considered a "correct" TR:

$$\boxed{P_{0t}^{L}=\frac{\sum p_t q_0}{\sum p_0 q_0}} \rightarrow \boxed{P_{t0}^{P}=\frac{\sum p_0 q_0}{\sum p_t q_0}} \rightarrow \boxed{P_{t0}^{L}=\frac{\sum p_0 q_t}{\sum p_t q_t}} \text{ or } \boxed{P_{0t}^{P}=\frac{\sum p_t q_t}{\sum p_0 q_t}} \rightarrow \boxed{P_{t0}^{L}=\frac{\sum p_0 q_t}{\sum p_t q_t}} \rightarrow \boxed{P_{t0}^{P}=\frac{\sum p_0 q_0}{\sum p_t q_0}}.$$

The first step, that is $p_0 \leftrightarrow p_t$ (or $P_{0t}^{L} \rightarrow P_{t0}^{P}$, and $P_{0t}^{P} \rightarrow P_{t0}^{L}$ respectively) without simultaneously interchanging quantities ($q_0 \leftrightarrow q_t$) is nowadays known as ***price reversal test***.[33] Evidently the index-pair Laspeyres-Paasche meets this test, because $P_{0t}^{L}P_{t0}^{P}=1$, and $P_{0t}^{P}P_{t0}^{L}=1$, but not TR in Fisher's definition of "time reversal") which requires also interchanging of quantities.

In my view, however, LvB's idea of time reversal as a two-stage process is not really convincing (and this applies – as will soon be seen – with even more force to a quite similar two-stage process LvB envisaged in his critique of the factor reversal test FR).

3. Insufficient reasoning and misplaced analogies

In the last analysis in LvB's view Fisher

a) rather arbitrarily gave priority to his reversal tests (TR and FR) over all other axioms ("*ziemlich willkürlich über alle anderen Kriterien emporgehoben*", B 1927, 751)[34]
b) based his predilection for such tests on clearly inappropriate analogies, viz.

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[31]. The derivation of the CSWD index of Carruthers, Selwood, Ward and Dalen follows (5) since the Harmonic mean of price relatives P^H and the arithmetic mean (i.e. Carli's index P^C) are time antithesis of one another.
[32] denoted by $P^{(TR)}$, $P^{(FR)}$ and $P^{(TFR)}$ in the relevant formulas below in Annex 2.
[33] See v.d.Lippe 2007, 208f. Its definition is $P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t)\, P(\mathbf{p}_t,\mathbf{q}_0,\mathbf{p}_0,\mathbf{q}_t) = 1$. It is a price reversal taken in isolation with quantities kept constant. To make this plausible imagine a household adapts itself to a new – maybe higher – price level by changing (reducing) its demand (quantities) only with a certain time lag in a second step.
[34] At that time "Kriterien" (criteria) was in Germany used for what now would rather be called "axioms". Note that "arbitrary" not only applies to the reversal tests themselves but primarily to giving priority to them.

- the *analogy "index - price relatives"* saying that a price index for n > 1 commodities should behave like a simple price relative $m_{i,0t} = p_{it}/p_{i0}$ for one commodity i only, where of course $m_{i,0t}$ satisfies TR as $m_{i,t0} = p_{i0}/p_{it} = (m_{i,0t})^{-1}$, [35] and
- an *analogy to justice (impartiality) or "fairness" and symmetry*, implicitly using the "*principle of insufficient reason*": there is no reason why what works forward $0 \rightarrow t$ should not work backward $0 \leftarrow t$ equally well, or (for FR) what applies to prices p_i should mutatis mutandis apply to quantities q_i too.

The assumption tacitly made here is that 0 and t (and therefore also P^L and P^P), or prices p and quantities q, are in all relevant aspects much the same, or so to say equal and "on the same level"; and that is precisely where LvB's and our criticism sets in.

b) More arguments against the TR

Sec. 2a presented three arguments advanced already by LvB against the TR as requirement a "good" index formula ought to fulfill. In what follows we add seven more which, despite being our own personal view, they may well be in the spirit of LvB, suggesting that Fisher's approach rests on some intuitive appeal only and on unwarranted symmetries, that is on :[36]

1. an unwarranted desirability of an index independent of the base year, or
2. the false equation, what applies to a single price should also apply to a price level,
3. a reversal of time is at odds with common experience, and having an "underlying order", as "time" typically has, is only another way of saying that a "reversal" is nonsense; conspicuously and not surprisingly therefore TR fits more interregional (with no underlying order) than intertemporal comparisons,[37]
4. the two periods, 0 and t in the TR test, are not periods of the same kind, and
5. for the TR test but not for real life it makes no difference whether 0 and t are point in time close to one another or widely separated
6. reversal tests, TR and FR foster the erroneous notion of a "bias" or a sort of mirror symmetry between the two allegedly equally well reasoned indices of Laspeyres and Paasche (what applies to P^L applies to P^P with opposite sign only), and
7. ironically and all too typically most (if not all) renowned systems of axioms do not mention TR (nor FR), as such reversal tests seem to be unduly restrictive.

For Fisher everywhere things seem to be on a par that rather should be kept distinct: 1 index numbers expressed in per cent (of a base value) and "absolute" figures (with no base), 2 prices and price levels, 3 comparisons across countries and across points or intervals in time, 4 we the base period 0 kept constant for some years and the is necessarily constantly changing actual period t.[38] In 5 we argue that TR may make sense with 0 and t close to one another but not when t is far away from 0. However if TR holds or fails, it does so for any 0 and t, no matter how close or distant. Finally in 7 we (once more) emphasize that reversal tests will in general be incompatible with other useful properties of index functions.

Ad 1: Even in really simple situations many people have difficulties with percentages.[39] We usually transform a series of absolute into relative figures in order to make them better com-

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[35] Analogy between indices and relatives was apparently the motivation for both, the TR and the FR.

[36] Some of the following arguments are merely illustrations of LvB's point 3b. This applies at least to arguments 1 through 4 against TR. All my arguments are taken from a referee report I wrote in 2013 about a paper submitted to the Journal of the Royal Statistical Society.

[37] Because time is irreversible: to suggest $0 \leftarrow t$ were as good as $0 \rightarrow t$ is not "fair" but simply off the track.

[38] In the sequence of indices P_{01}, P_{02}, P_{03} we have only one base (0) but t takes on three values 1, 2, and 3.

[39] I remember how journalists reporting on a lawsuit (about the pay and allowance system for civil servants) at the German Constitutional Court in Karlsruhe could hardly understand why living in Bavaria was not by 20% - as it appeared "logical" to them at first glance (but only 16.7%) cheaper than Munich when prices in Munich are 20% higher than in Bavaria. In a similar vein it is difficult for many to imagine that things have *changed* (to the

parable. What enhances comparability is just *the common reference to the same base* because each figure is expressed relative to, or "in units of" just this very base (which, however, often is not made explicit). So with an index the choice of the base necessarily matters, and attempts to circumvent this problem, for example with chain indices or by requiring a strict relation between two bases, 0 and t, like $P_{t0} = (P_{0t})^{-1}$ rather indicate that index numbers are not well understood.[40] The problem of choosing the "correct" base or failing TR should be accepted as price for the better comparability of relative as opposed to absolute numbers.[41]

Ad 2: Given that quite a few people have problems with percentages and must have experienced that relying on intuition we not always get things right it should be not too annoying, that things may not be so easy with a *price level* of *many* commodities as they are with the *price of a single good*. A significant difference between the one-good and the many-goods situation is that we can handle the latter situation in two different ways: forming a ratio of averages (ROA) or an average of ratios (AOR).[42] With a single price a ratio of prices p_{it}/p_{i0} *is* a price ratio (relative) and also a ratio $\bar{p}_t/\bar{p}_0$. The idea of the time reversal test goes back to the Dutch economist N. G. Pierson (1896) who drew attention to an inconsistency between ROA and AOR in the case of unweighted indices.[43] Some (alleged) ambiguities of Carli's index[44] $P_{0t}^C = \frac{1}{n}\sum(p_{it}/p_{i0})$, deemed so severe to him "that the system of index numbers is untrustworthy" (p. 130) and "is not to be reconstructed, but to be abandoned altogether" (p. 127).

He studied a fictitious numerical example (tab. 2) with two commodities in three situations and compared price indices of Carli P_{0t}^C, Jevons $P_{0t}^J = \prod(p_{it}/p_{i0})^{1/n}$, and Dutot $P_{0t}^D = \bar{p}_t/\bar{p}_0$:

Table 2: Numerical example of N. G. Pierson

	situation I		situation II		situation III		situation II*	
	i = 1	i = 2	i = 1	i = 2	i = 1	i = 2	i = 1	i = 2
p_{i0}	50	100	100	100	50	200	200	50
p_{it}	100	50	200	50	100	100	100	100
Carli P^C	1.25		1.25		1.25		1.25	
Jevons P^J	1		1		1		1	
Dutot P^D	150/150 = 1		250/200 = 1.25		200/250 = 0.8		200/250= 0.8	

Pierson favorite index was apparently P^D (a ROA formula). He gave, however, no reasons why P^D should be preferred over an AOR approach like P^C or P^J. Situation III can be viewed

worse) after a decline by 20% and a subsequent rise by 20% or vice versa (they intuitively tend to think - 20% is canceled by + 20%). The problem with percentages obviously is that in quoting such figures we notoriously forget what is meant by 100%. Clearly the same additional X (in absolute terms) will entail a higher percentage with base "Bavaria = 100" than with "Munich = 100" (as Munich has a higher price level than Bavaria).

40 Moreover it is a misunderstanding on an incredibly low level of Statistics at that: many elementary text books demonstrate at length with numerical examples that a given time series in *absolute* figures x_1, x_2, … will generate quite different time series of index numbers (necessarily *relative* by their nature) depending on which of the x's serves as base (as if we should get the same graph with each x as base). Worse even: what is merely trivial and unavoidable, i.e. that the base matters, is often dramatized to an ostensibly severe defect of index numbers.

41 Irrelevance of the base is not in itself favourable and worthwhile to be aimed at. It is sometimes said that chain indices have no base as it is constantly updated and always just the preceding year. However, this applies to the factors, or links $P_{t-1,t}$ only. Characteristic for chain indices is the existence of many links with *many bases* t - 1 (t = 1, 2,…) *multiplied* to form a chain. The focus is not on links but on their product (i.e. the chain indices), and there is neither "no base" nor a *common reference to the same base* (usually seen as major advantage of indices).

42 There is no such duplicity in the single-good-case, where of course from $p_{it}/p_{i0} = 1{,}2$ follows that $p_{i0}/p_{it} = 1/1.2 = 0.833$ (that is -16.7%) for this only good i. But to what refers 20% or 16.7% in the many-goods-case: to the change of an average price or to an average of the various changes of prices? The difference between ROA and AOR of course vanishes when each price changes at the same rate $p_{it}/p_{i0} = \lambda, \forall$ i.

43 Note that only Jevons' index P^J (out of the indices above) allows both interpretations, ROA *and* AOR.

44 or Sauerbeck's index as it then was called. It is of course questionable whether "ambiguities" exist here at all.

as time reversal of situation II (strictly speaking, the time reversed II is rather II*, but none of the indices reflects the difference between III and II*)[45], correctly reflected in $P^J = 1 = 1^{-1}$ but much more eye-catching in P^D where indeed $0.8 = (1.25)^{-1}$. By contrast P^C is clearly inadequate for him as it fails to make a difference between the two situations. From the point of view of TR among the indices P^D and P^J one index should be as good as the other, yet Pierson rejected P^J, because the "geometrical method … leaves the average price unaltered in each of these cases, which is clearly a mistake" (p. 130). By this he obviously meant that the result $P^J = 1$ is independent of the absolute levels $\overline{p}_0$ and $\overline{p}_t$ respectively so that P^J (and P^C) treats I and II alike.[46] Interestingly though index numbers are *relative* figures the case for TR makes recourse to *absolute* prices (and quantities): TR requires that p_{i0} is interchanged with p_{it} (not that $p_{1t}/p_{10} = 2$ is set off by $p_{2t}/p_{20} = p_{10}/p_{1t} = ½$ as for example in I). The focus is also laid on absolute prices when Pierson argues that situations I, II and III should be treated differently despite identical price relatives, $p_{1t}/p_{10} = 2$ and $p_{2t}/p_{20} = ½$ in all three cases.[47]

It may be noteworthy that Pierson rightly realized that the difference between P^C (an index he disapproved) and P^D depends on the base period prices. P^C gives a price relative less weight than P^D when $p_{i0}/\Sigma p_{i0} > 1/n$ and more when $p_{i0}/\Sigma p_{i0} < 1/n$, because $P^D_{0t} = \sum \frac{p_{it}}{p_{i0}} \cdot \frac{p_{i0}}{\sum p_{i0}}$.[48]

In sum there are significant differences between the one-good and the many-goods situation in that there now, in the latter case alternatives unknown to the one-good case emerge, such as the dichotomy ROA vs. AOR, or different conclusions we may reach at depending on whether our emphasize is on absolute prices or on price relatives. Hence simple analogies between the one-good and the many-goods case are misplaced and inappropriate and we rather should be prepared to get (and accept) a different picture of a process when looking at it backward from t, that is with the $t \rightarrow 0$ perspective instead of forward (from a $0 \rightarrow t$ perspective).[49]

Ad 3: Time is usually visualized as arrow with a clear distinction between cause (C) and effect (E) or "before" (C) and "after" (E). What happens *after* E cannot be the cause of E. We *remember* the known and definite past but can only *expect* a necessarily "open" and indeterminate future. There is no point in making assumptions or forming expectations about the past. So C and E are clearly *different phenomena* that *deserve to be treated differently*. It is pointless to *both* perspectives $C \rightarrow E$ *and* $E \rightarrow C$ (only either $C \rightarrow E$ *or* $E \rightarrow C$ makes sense.

In physics it is the increase of "entropy" (disorder) that gives the flow of time a direction. We see a cup of water (an object of high order) falling off a table and breaking into pieces, but it is most unlikely (though not *logically* impossible) to see these pieces jumping back to the ta-

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45 Relating p_{jt} to p_{i0} and p_{it} to p_{j0} rather than p_{it} to p_{i0} and p_{jt} to p_{j0} makes no difference for TR.

46 For both indices, P^J and P^C, only counts that in all four situations we have the same two relatives amounting to 2 and ½ (no matter to which of the two goods each price relative belongs). Absolute prices are irrelevant. This and equality of III and II* is not always the case. Remember, Pierson only studied unweighted indices. He saw, however, already the problem of commensurability with P^D and he proposed (like Drobisch) to use price quotations only that refer to the same unit of quantity (a hundredweight for example). This might be applicable – if at all – to some commodities but definitely not to services.

47 That III is in a way a "reversed" II is also owed to our orientation on absolute prices, and the focus is also on absolute rather than relative prices when Pierson argues that P^D should rank higher than P^J or P^C.

48 Pierson also correctly saw that the plausibility of the argument Laspeyres advanced in support of the arithmetic mean in his dispute with Jevons (and against Jevons' geometric mean) – which, however, was an argument pro Dutot P^D rather than pro P^C– is owed to the fact that Laspeyres had chosen equal base period prices (like in II situation with $p_{10} = p_{20} = 100$); with unequal prices p_{i0} we no longer have $P^C = P^D$ (v.d.Lippe 2013, 341).

49 This should be acceptable as everybody knows that situations 0 and t as a rule differ as regards availability and relative importance of goods. And it holds all the more when the past (0) is long ago, say 20 or 30 years. We maintain (in line with our fifth argument): to require (as done in TR) an *exact* functional relation between P_{t0} an P_{0t} *for any difference between 0 and t,* however long it may be, is clearly pointless. TR may – at best – be desirable when (in t) 0 is not far ago.

ble and recollecting again to a well formed glass of water. Wave propagation *starts at a source*, we never see a wave travelling back and *ending at its source* where it is absorbed. Heat emission (radiation) goes from hot to cold, not the other way round. Because time has an inherent order (sequence) $t_1 \rightarrow t_2$ the two points t_1 and t_2 are not staying interchangeably on the same level. A reversal to $t_2 \rightarrow t_1$ would be counter-intuitive and anything but an embodiment of "fairness".

A reversal makes sense, however, with two countries; say A (Austria) and F (France). It does so just because there is no inherent natural order between countries (there is no reason to prefer A to F, or F to A), and for just the same reason it is desirable to have a unique purchasing power parity (PPP) that is P_{AF} (base county A) should be unequivocally related to P_{FA} (base country F) as for example $P_{AF} = (P_{FA})^{-1}$.[50] That countries are unordered also makes it desirable, to have transitivity, that is a consistent order (sequence) of all countries in the one dimension of PPP.[51] So it is not by coincidence that LvB demonstrated the in-transitivity of the "chain system" (and its disadvantage) with the following example[52]

> **Ex. 2:** Assuming four countries and Fisher's index P^F LvB explicitly realized that a "chaining" A-B-C-D will as a rule differ from A-C-B-D (and of course from A-D), $P^F_{AB}P^F_{BC}P^F_{CD} \neq P^F_{AC}P^F_{CB}P^F_{BD}$. Multiplication (chaining) of binary indices is ambiguous "unless a choice is made once and for all for one and only one definite sequence of places".[53]

While we can make *any indirect* comparisons across countries are *equivalent* it is uncommon to make other indirect comparisons between points in time than by way of time series *where only* $t_1 < t_2 < t_3$…. is in accordance with the *definite order* of time. It would be queer to study a sequence t_3, t_1, t_5, t_4, t_2 or so. Thus existence of a natural order is tantamount to make reversibility meaningless. Reversibility is not a question of "fairness", it is rather a misled concept when applied to time.

Ad 4: The TR idea ignores that *0 and t are periods of different kind*. We are used to keep 0 in P_{0t} constant for a couple of years[54], whereas t in P_{0t} strictly speaking denotes *a number of* periods (P_{01}, P_{02}, …), not just one period. Fisher was wrong in not accounting for this conceptual difference between a (temporarily) *constant* base period 0 and a constantly *varying* period t. There is no point in interchanging 0 and t, not only because time is an arrow ($0 \rightarrow t$ exists, not $0 \leftarrow t$) but also because periods 0 and t serve different purposes in index numbers.

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[50] Country reversibility (CR), as analogon of time reversibility (TR) only makes sense because there is no natural order so that $A \rightarrow F$ or $F \rightarrow A$ makes no difference, or – put differently – *here*, with countries it makes sense to treat things symmetrically. However, this may no longer apply when weights are involved, as for example a "basket of consumer goods" in the case of consumer price indices. It is not unreasonable to insist on using the own country's basket (there are no good reasons not to use the basket of A in P_{AF}), and then nobody expects P_{AF} (where we use the Austrian "basket") to be prima facie somehow related to P_{FA} (on the basis of French household expenditure data). Such expectations were at best justified when in both cases the same "basket" is used.

[51] Transitivity requires consistency between the direct *and all indirect comparisons* between *any two countries*. It is just because countries are unordered that there is no reason to prefer one indirect comparison over another.

[52] He concluded that there is no unequivocal result "*oder es müsste zuvor ein für alle Male das Prinzip, nach welchem man eine Anzahl miteinander in Bezug auf das durchschnittliche Preisniveau zu vergleichende Orte anordnen soll, eigens festgelegt werden*" B 1924/II, 219. Such a choice of a unique sequence in a series of indirect comparisons is made in the method of a "minimum spanning tree" (in the framework of international comparisons). See for this method v.d.Lippe 2007, 525. This amounts to saying – as said above – that the chain index makes use of chaining but is not chainable (transitive); v. d. Lippe 2001, 35; and v. d. Lippe 2007, 467.

[53] "No order" between countries say A, B, C and D with fixed A and D (when A and D are compared) is by definition equivalent to saying that there should be no difference between the 2! = 2 permutations of B and C. Not only are there different indirect comparisons, the different empirical results can hardly be explained.

[54] I know of course the problem (raised in the chain index discussion) of how many years is "a couple".

Ad 5: For the TR it makes no difference which is the period, 0 and t. Once an index formula satisfies TR, this *holds for any two periods 0 and t however apart they may be from one another* (provided all relevant data $\mathbf{p}_0$, $\mathbf{p}_t$, and $\mathbf{q}_0$, $\mathbf{q}_t$ are available). However, economically it is highly relevant which periods, 0 and t we refer to, in particular how distant t is from 0. We often hear that weights less frequently than annually updated as no longer "relevant" or "representative" (and therefore a chain index is needed) because nowadays progress is so fast that to compare 2010 to 2015 is like comparing 1900 to 1950 had been in former days. With this in mind, it is strange require TR, since if TR holds it holds for *any two periods,*[55] for $P_{2015,2100} = (P_{2010,2150})^{-1}$ where it might be reasonable, as well as for $P_{1980,1900} = (P_{1900,1980})^{-1}$ (in 1900 we not yet had airplanes while in 1950 it was not unusual to fly to Madeira or so for holydays).

Ad 6: Fisher referred to his reversal tests when he introduced *the notion of a "bias" of an index* (notably a context in which no sampling is involved). He endorsed – as his followers continue to do – the wrong idea that P^L and P^P are equally well justified[56] only with an opposite sign of the bias (Laspeyres is "biased" upwards because $P^L_{0t}P^L_{t0} > 1$ and $P^L_{0t}Q^L_{0t} > V_{0t}$ just like Paasche is biased downwards), so that this could best be cancelled out by crossing. Hence for Fisher reversal tests also serve as argument for the "ideal index" of Fisher[57] $P^F_{0t} = \sqrt{P^L_{0t}P^P_{0t}}$. It is a myth, however, that P^L and P^P have an equally well established rationale:

- for P^L inflation takes place to the extent that *to buy the same quantities* (or basket or vector $\mathbf{q}_0$) will be more expensive, or $\Sigma p_t q_0$ departs from $\Sigma p_0 q_0$, so that elements in a P^L-series differ with respect to prices only $P^L_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0}$, $P^L_{02} = \frac{\sum p_2 q_0}{\sum p_0 q_0}$,…[58]
- while successive elements in a P^P-series $P^P_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_1}$, $P^P_{02} = \frac{\sum p_2 q_2}{\sum p_0 q_2}$,… differ with respect to both, prices and quantities and inflation takes place to the extent that *values* (at current prices, *nominal expenditures*, the numerators in the P^P indices) increasingly exceed *volumes* (at constant prices, *real expenditures*, denominators in the P^P indices)

Accordingly just as P^L used at least to serve primarily for inflation measurement so P^P to deflate (i.e. translate "values" into "volumes") aggregates of National Accounts.[59]

Ad 7: A predilection for TR must be viewed against the backdrop that TR may well rule out many useful index functions. Even if TR were reasonable as such it should not be achieved at the expense of other reasonable properties or of violating other reasonable axioms. In all inconsistency theorems ("there is no index function that…") I know of either or both, circularity and reversibility (TR, FR) is involved (v.d.Lippe 2007, 184, 215). Also LvB already noticed that Drobisch's unit-value-index P^{DR} while passing the circular and TR test violates identity. Typically enough renowned systems of axioms such as the Eichhorn and Voeller system

[55] TR is usually achieved by making use of both "baskets" (vectors of quantities q, or "weighting schemes"), not only of $\mathbf{q}_t$, but also of $\mathbf{q}_0$. We can't convincingly argue against fixed basket (fixed weighted) indices like P^L in favour of chain indices saying that $\mathbf{q}_0$ becomes progressively irrelevant and unrepresentative (as t > 0) and at the same time continually use $\mathbf{q}_t$ in addition to $\mathbf{q}_0$. In other words, I think it is a bit contradictory to advocate on the one hand chain indices (because of rapid changes of consumption patterns) and to require TR on the other hand.

[56] "Nothing can be offered in proof of the superiority of the one over the other" (Walsh as discussant in Fisher 1921; 538). In my view because of reasons given above this statement, though popular and frequently used to justify ones preference for Fisher's ideal index P^F is bluntly wrong,

[57] A formula - according to B 1924, 851 - already brought into play by Bowley (1899) and Walsh (1901). This is undisputed; see also Fisher 1921. Yet to call P^F "Fisher's index" has since long grown into a habit.

[58] This is meant by "pure price comparison" (fulfilled by P^L but not by P^P and much less by chain indices).

[59] In my view the only reasonable argument in favour of the FR test (and not only the less strict product test) is that the same index P can serve both purposes, inflation measurement and deflation.

(v.d.Lippe 2007, 220) usually do not mention reversal tests but instead preferred much less appealing ideas, such as "linear homogeneity" for example. So TR/FR must be dispensable.

To sum up: LvB is right in saying that reversal tests are badly reasoned as they invoke categories like symmetry and justice (in the sense of what is true for 0 is true for t).[60] To suggest equality of periods 0 and t or –worse even – p and q (thus also indices, P_{0t} and Q_{0t}) is not an embodiment of "fairness", it is simply erroneous. Reversal tests should be abandoned. There is no use in thinking in analogies and treating different things alike in index theory.

3. Factor reversal test (FR) and Bortkiewicz's criticism of Fisher's "ideal index"

Here LvB again had two-stages (prices first, and then adjusting quantities) in mind:

$\boxed{\sum p_0 q_0}$ step 1 $\boxed{\sum p_t q_0}$ step 2 $\boxed{\sum p_t q_t}$ and in terms of index functions

$$\boxed{P^L_{00} = \frac{\sum p_0 q_0}{\sum p_0 q_0}} \rightarrow \underset{\boxed{P^L_{0t}}}{\boxed{P^L_{0t} = \frac{\sum p_t q_0}{\sum p_0 q_0}}} \rightarrow \underset{\boxed{Q^P_{0t}}}{\boxed{Q^P_{0t} = \frac{\sum p_t q_t}{\sum p_t q_0}}},$$

which yields $P^L_{0t} Q^P_{0t} = V_{0t}$ (to explain the equally valid relation $P^P_{0t} Q^L_{0t} = V_{0t}$ seems to be even less straightforward).

Interestingly to consider changes in the variables in a *two step operation* (one by one) appears to justify using P^L and Q^P, that is confining oneself to the less demanding "product test".[61] There is no reason to require FR as an interchange of index types $P_{0t} \leftrightarrow Q_{0t}$ by interchanging variables $p \leftrightarrow q$ in only one step so that $\boxed{P^L_{0t} = \sum p_t q_0 / \sum p_0 q_0} \rightarrow \boxed{Q^L_{0t} = \sum q_t p_0 / \sum q_0 p_0}$ and $\boxed{P^P_{0t}} \rightarrow \boxed{Q^P_{0t}}$ as in: $P^F_{0t} = \sqrt{\frac{\sum p_t q_t}{\sum p_0 q_t} \frac{\sum p_t q_0}{\sum p_0 q_0}} = \sqrt{P^P_{0t} P^L_{0t}} \rightarrow Q^F_{0t} = \sqrt{\frac{\sum q_t p_t}{\sum q_0 p_t} \frac{\sum q_t p_0}{\sum q_0 p_0}} = \sqrt{Q^P_{0t} Q^L_{0t}}$.

LvB's idea of two stages seems less convincing, here in the case of FR than in the TR case where we saw that in both sequences $P^L_{0t} \rightarrow P^P_{t0} \rightarrow P^L_{t0}$, and $P^P_{0t} \rightarrow P^L_{t0} \rightarrow P^P_{t0}$ we *first* interchanged (reverses) *prices* $p_0 \leftrightarrow p_t$ *and then quantities* $q_0 \leftrightarrow q_t$. However, a similar two-stages-process to visualize FR seems to be pretty farfetched. In $P^L_{00} \rightarrow P^L_{0t} \rightarrow Q^P_{0t}$ as above, the first step $p_0 \rightarrow p_t$ only affects the numerator in P^L, and in the second step (or $P^L_{0t} \rightarrow Q^P_{0t}$) a quantity change $q_0 \rightarrow q_t$ takes place, but only in the numerator, and in addition we have one more (as in the first step) a change $p_0 \rightarrow p_t$ (again only partial, i.e. now only in the denominator).

With $P^P_{tt} \rightarrow P^P_{0t} \rightarrow Q^L_{0t}$ (to motivate $P^P_{tt} Q^L_{0t} = V_{0t}$) we also would run into difficulties, let alone with assuming a price change *after* the quantity change,[62] as for example in $Q^L_{00} \rightarrow Q^L_{0t} \rightarrow P^P_{0t}$. So to imagine a two-step-procedure is far from convincing when not only two points in time ($0 \leftrightarrow t$) but two different variables ($p \leftrightarrow q$) and two types of index functions (P_{0t} and Q_{0t}) are involved. Moreover, in the case of FR not only the idea of two stages, also the idea of "symmetry" is not plausible (here much less even than it was already in the TR context). Reasoning in terms of "symmetry" is inappropriate when different variables such as p and q are involved.

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60 instead of claiming for example, that an index only provides certain insights if it meets the reversal tests.

61 Fisher was – in my and certainly also LvB's view – unable to give compelling reasons for why we should make use of the much more restrictive (and thus rarely satisfied) factor reversal test rather than the less demanding "product test" (a sort of "weak" factor reversal test) which not requires that both indices, P and Q, need to have the same formal structure. As aforesaid (footnote 59) it appears to me that the only reasonable argument for the FR test is that the same price index P can serve inflation indicator *and* deflator.

62 This clearly would be economically, i.e. from the point of view of consumer behavior, much less plausible. Note that for TR there was no need to consider the reverse sequence, first $q_0 \rightarrow q_t$ and then $p_t \rightarrow p_0$.

To mention only one problem: it is for example in general admissible to summate over n prices ($\Sigma_i p_{it}$, i = 1,2,…,n) but often not reasonable, if not impossible to consider sums ($\Sigma_i q_{it}$).[63]

As to LvB's other objections to this reversal test we may simply note that most of what he said against TR for him applies mutatis mutandis to FR as well: this axiom too is insufficiently reasoned, based on inappropriate analogies and allusions to "fairness" and "symmetry". Again compliance with FR as with all reversal tests can easily be ensured in a purely formal manner (in this case by crossing P with its "factor antithesis" V/Q, see Annex 2).

Referring to the ***"ideal index"*** $P^F = (P^L P^P)^{1/2}$ it seems that LvB endorsed all or most of the then in Germany widespread objections against Fisher's purely formal or "formalistic" approach and P^F, a formula generally at that time in Germany, not only by LvB considered as formula with no "economic content".[64]

It may be useful to emphasize one final point important for LvB and come back to the dichotomy AOR (average of relatives) and ROA (ratio of averages). An index like P^L may possess a "double interpretation" as AOR (with weights $w_i = p_{i0}q_{i0}/\Sigma p_{i0}q_{i0}$) and also as ROA (ratio of expenditures with quantities q_{i0} held constant), a property LvB called "two-way" interpretation *("Zwieförmigkeitskriterium")*[65]; for him a most important quality indicator of an index, that deserves to be upgraded to an axiom or "test" (with more justification than TR and FR). Consequently he criticized Fisher's "ideal index" P^F for not allowing any of the two interpretations (neither AOR nor ROA,).[66]

Annex

1. Bortkiewicz's (general) theorem on linear index functions as originally derived in 1923

In what follows we simply quote LvB's theorem (what he called "Schema M" [M scheme], where M possibly stands for "moments" as a term in statistics) using his original notation and we contrast his symbols with our symbols (as they are used throughout in this paper and also in v.d.Lippe 2007):

Bortkiewicz's symbols	Our symbols
a (price relatives), w (expenditure weights)	$a = p_t/p_0$, $w = p_0q_0/\Sigma p_0q_0$, $m_1 = \Sigma wa/\Sigma w = P^L_{0t}$
b (quantity relatives), $m_2 = \Sigma wb/\Sigma w$	$a = p_t/p_0$, $m_2 = Q^L_{0t}$
$m_1' = \sum a(bw)/\sum bw$	$m_1' = P^P_{0t}$
$m_2' = \sum b(aw)/\sum aw$	$m_2' = Q^P_{0t}$
$r\sigma_1\sigma_2 = \sum w(a-m_1)(b-m_2)/\sum w$	$C = \sum p_0q_0(p_t/p_0 - P^L_{0t})(q_t/q_0 - Q^L_{0t})/\sum p_0q_0$
$m_2' - m_2 = r\sigma_1\sigma_2/m_1$	$Q^P_{0t} - Q^L_{0t} = C/P^L_{0t}$
$m_1' - m_1 = r\sigma_1\sigma_2/m_2 = h/m_2$	$P^P_{0t} - P^L_{0t} = C/Q^L_{0t}$

[63] Quantities of different goods often require different units of measurement but even with the same units to form a sum can be nonsense. What is the meaning of 2 liters (of what?) when we add 1 litre milk and 1 litre petrol?

[64] It was only much later that it became common practice to argue also in favour of P^F by referring to the so called "economic theory of index numbers" (i.e. in the microeconomic framework of utility-maximization).

[65] See von B 1927, 747. LvB there also made a distinction between a simple (*einfaches*) and a qualified (*qualifiziertes)* "two-forms-criterion"(*Zwieförmigkeitskriterium*). The latter requires economically motivated weights (*daß die Gewichte …materiell motiviert seien*); He criticized P^J for not passing the simple criterion (the qualified criterion of course does not apply as there are no weights in P^J). I think this cannot hold water: P^J clearly is AOR but it can also be written as ratio of "geometric mean prices" p_{it} (numerator) and p_{i0} (denominator)

[66] ROA perhaps in a quite farfetched manner, when for example $(\Sigma p_tq_t\Sigma p_tq_0)^{1/2}$ and $(\Sigma p_0q_t\Sigma p_0q_0)^{1/2}$ are viewed as some fictitious expenditures It may be interesting to quote here Allyn Young, 1923; 359 (a contemporary of LvB): "In a way Professor Fisher is right in holding that all true index numbers are averages of ratios.' But I should prefer to say that all true index numbers are at once averages of ratios and ratios of aggregates."

2. How to construct formulas that comply with TR, FR, and both "Reversal Tests"?

A short overview of the following formulas is given in B1924, 850.

a) TR (how to generate a time reversible index)

For a time reversible price index corresponding to P_{0t} take the geometric mean

(6) $$P_{0t}^{(TR)} = \sqrt{P_{0t}\frac{1}{P_{t0}}}$$

of P_{0t} and $(P_{t0})^{-1}$. The rationale of this formula is explained above. Evidently $P_{0t}^{(TR)}P_{t0}^{(TR)} = 1$, and with $P_{0t} = P_{0t}^{L}$, and $P_{t0}^{L} = 1/P_{0t}^{P}$, or $P_{0t} = P_{0t}^{P}$ and $P_{t0}^{P} = 1/P_{0t}^{L}$ we get $P_{0t}^{(TR)} = P_{0t}^{F}$.[67]

b) FR (how to generate afactor reversible index)

$P_{0t}^{(FR)}$ is a factor reversible price index of P_{0t} in combination with Q_{0t}, the quantity index that corresponds to P_{0t} (or in Fisher' s words V_{0t}/Q_{0t} is the "factor antithesis" of P_{0t}) if:

(7) $$P_{0t}^{(FR)} = \sqrt{P_{0t}\frac{V_{0t}}{Q_{0t}}}$$, where V_{0t} is the value index $\Sigma p_t q_t/\Sigma p_0 q_0$ (or value ratio), and

(7a) $$Q_{0t}^{(FR)} = \sqrt{Q_{0t}\frac{V_{0t}}{P_{0t}}}$$, is the corresponding factor reversible quantity index

and it can easily be seen that $P_{0t}^{(FR)}Q_{0t}^{(FR)} = \sqrt{(V_{0t})^2} = V_{0t}$ while in general $P_{0t}Q_{0t} \neq V_{0t}$.

Again we may apply the formula $P_{0t}^{(FR)}$ quite mechanically and end up with an index devoid of any meaning. For example for the Walsh index $P_{0t}^{W} = \sum p_t\sqrt{q_0 q_t}\big/\sum p_0\sqrt{q_0 q_t}$ and Q_{0t}^{W} gained by interchanging p's and q's the corresponding factor reversible price index would be $P_{0t}^{(FR)W} = \sqrt{P_{0t}^{W}(V_{0t}/Q_{0t}^{W})} = \sqrt{P_{0t}^{W}Q_{t0}^{W}V_{0t}}$ since P^W and Q^W are time reversible, or spelled out $P_{0t}^{(FR)W} = \sqrt{\frac{\sum p_t\sqrt{q_0 q_t}}{\sum p_0\sqrt{q_0 q_t}}\frac{\sum q_0\sqrt{p_0 p_t}}{\sum q_t\sqrt{p_0 p_t}}\frac{\sum p_t q_t}{\sum p_0 q_0}}$. The quantity counterpart $Q_{0t}^{(FR)W}$ is $\sqrt{Q_{0t}^{W}P_{t0}^{W}V_{0t}}$.

c) TFR (how to generate indices satisfying both reversal tests)

Substituting $P_{0t}^{(FR)}$ and $P_{t0}^{(FR)}$ for P_{0t} and P_{t0} in (6) gives a formula passing both reversal tests,

(8) $$P_{0t}^{(TFR)} = \sqrt[4]{(V_{0t})^2\frac{P_{0t}Q_{t0}}{P_{t0}Q_{0t}}} = \sqrt{V_{0t}}\cdot\sqrt[4]{\frac{P_{0t}Q_{t0}}{P_{t0}Q_{0t}}}$$ and for the corresponding quantity index

$Q_{0t}^{(TFR)}$ we get $Q_{0t}^{(TFR)} = \sqrt{V_{0t}}\cdot\sqrt[4]{\frac{Q_{0t}P_{t0}}{Q_{t0}P_{0t}}}$ (B 1923/I,387).[68]

To try an index which is neither time nor factor reversible, we take for example Carli's index $P_{0t}^{C} = \frac{1}{n}\sum\frac{p_t}{p_0}$. A *Carli type time reversible index* would be $P_{0t}^{(TR)C} = \sqrt{\frac{\sum(p_t/p_0)}{\sum(p_0/p_t)}} = \sqrt{P_{0t}^{C}P_{0t}^{H}}$, where P_{0t}^{H} is the harmonic mean, the time antithesis of P^C. This index formula $P^{(TR)C}$ is known

[67] because P^L (Laspeyres) is the time antithesis of P^P (Paasche) and vice versa. Likewise P^L is the factor antithesis of Q^P (and P^P of Q^L) so that P^F and Q^F as geometric means or "rectified" indices are factor reversible.

[68] With Walsh indices we not surprisingly get the same formulas as before, that is $P^{(TFR)W}=P^{(FR)W}$ and $P^{(TFR)W} = P^{(FR)W}$, because the Walsh index is time reversible. Also not surprisingly with the Laspeyres or Paasche formula for P_{0t} and Q_{0t} the formulas 6 through 8 lead us to Fisher's "ideal" index formulas, P^F and Q^F respectively.

as the (time reversible) CSWD-price-index,[69] and the CSWD quantity index then is $Q_{0t}^{CSWD} = \sqrt{\frac{\sum(q_t/q_0)}{\sum(p_t/p_0)}} = \sqrt{Q_{0t}^{C} Q_{0t}^{H}}$. To find a *Carli-type factor reversible index* using (7) gives us $P_{0t}^{(FR)C} = \sqrt{\frac{\sum(p_t/p_0)}{\sum(q_t/q_0)} \frac{\sum p_t q_t}{\sum p_0 q_0}} = \sqrt{V_{0t} \frac{P_{0t}^{C}}{Q_{0t}^{C}}}$ and $Q_{0t}^{(FR)C} = \sqrt{\frac{\sum(q_t/q_0)}{\sum(p_t/p_0)} \frac{\sum p_t q_t}{\sum p_0 q_0}} = \sqrt{V_{0t} \frac{Q_{0t}^{C}}{P_{0t}^{C}}}$, so that evidently $P_{0t}^{(FR)C} Q_{0t}^{(FR)C} = V_{0t}$. Finally (8) permits the construction of a rectified or *time <u>and</u> factor reversible* index on the basis of Carli's index as follows

$$P_{0t}^{(TFR)C} = \sqrt{V_{0t}} \cdot \sqrt[4]{\frac{P_{0t}Q_{t0}}{P_{t0}Q_{0t}}} = \sqrt{V_{0t}} \cdot \sqrt[4]{\frac{\left(P_{0t}^{CSWD}\right)^2}{\left(Q_{0t}^{CSWD}\right)^2}} = \sqrt{V_{0t} \frac{P_{0t}^{CSWD}}{Q_{0t}^{CSWD}}} \text{ and } Q_{0t}^{(TRFR)C} = \sqrt{V_{0t} \frac{Q_{0t}^{CSWD}}{P_{0t}^{CSWD}}}.$$

Clearly $P_{0t}^{(TFR)C} Q_{0t}^{(TFR)C} = V_{0t}$ and since V_{0t} and the (P and Q type) CSWD indices are time reversible, so is the pair $P^{(TFR)C}$ and $Q^{(TFR)C}$. However, to find some plausibility or economic rationale for using such somewhat awkward index formulas like $P_{0t}^{(TFR)C} = \sqrt{V_{0t} \frac{\sqrt{P_{0t}^{C} P_{0t}^{H}}}{\sqrt{Q_{0t}^{C} Q_{0t}^{H}}}}$ as a price index and $Q_{0t}^{(TFR)C}$ as a quantity index must be pretty challenging.. Already LvB expressed doubts about the price-index character of a "rectified" $P_{0t}^{(TFR)}$ index formula and the quantity index character of a $Q_{0t}^{(TFR)}$ formula (B 1923/I, 393).[70]

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[69] As acronym for Carruthers, Selwood, Ward and Dalen. It is clear that this index can be gained by crossing (here taking a geometric mean of P^C and P^H. With "rectifying" we then have – in Fisher's terminology – a case of "double crossing" with a fourth root (geometric mean of geometric means). P^{CSWD} is also known to provide an approximation to Jevons' index, the unweighted geometric mean of price relatives (v.d.Lippe 2013, 349).

[70] It is not obvious why $(V_{0t}P_{0t}/Q_{0t})^{1/2}$ is a price index as opposed to the quantity index $(V_{0t}Q_{0t}/P_{0t})^{1/2}$.

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