Problems of operationalizing the concept of a "cost-of-living-index"

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Abstract

The aim of our paper is to discuss the problems of operationalizing the concept of a "cost-of-living-index" (COLI). For this purpose we are first undertaking a theoretical analysis of Diewert's theory of superlative index numbers as one possible approach to approximate a COLI. We show that Diewert's superlative index approach is arguable in many points and that the approach requires restrictive assumptions which are not likely to be met in observed households' behaviour. To get a better idea about the deviation of observed households' behaviour from the neoclassical assumptions about utility maximizing behaviour, we are estimating an Almost Ideal Demand System and a Quadratic Almost Ideal Demand System with cross section micro data from the German income and expenditure survey. Using the results of the demand system estimations we calculate COLIs and compare them with superlative index numbers and the Laspeyres price index.

1 Introduction

In some countries, especially the USA the prefered target of inflation measurement is the so called "cost-of-living-index" (COLI, or index of Konüs P_{0t}^{K}) as opposed to the traditional approach to define a consumer price index (CPI) which is based on a constant "basket" of goods (also known as cost of goods index, or COGI) as for example the well known Laspeyres price index. The COLI is the ratio of minimum costs c required to attain the same utility level $\bar{u}$ under two "price regimes" given by the price vectors $\mathbf{p}_t$ and $\mathbf{p}_0$ respectively, that is

$$P_{0t}^{K}\left(\mathbf{p}_0, \mathbf{p}_t, u\left(\mathbf{q}\right), \bar{u}\right) = \frac{c\left(\mathbf{p}_t, \bar{u}\right)}{c\left(\mathbf{p}_0, \bar{u}\right)}, \tag{1}$$

The COLI thus compares the expenditures required for attaining the same level of utility (well-being) rather than those required for buying the same quantities of goods. In the COGI approach quantities $\mathbf{q}_0$ and $\mathbf{q}_t$ are regarded as exogenous (i.e. independent of prices) and given, as e.g. in the Laspeyres price index

$$P_{0t}^{L} = \frac{\mathbf{p}_t'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0} \tag{2}$$

or the Paasche price index

$$P_{0t}^{P} = \frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_0'\mathbf{q}_t} \tag{3}$$

whereas in the COLI context, $\mathbf{q}$-vectors are said to be determined or "explained" by (rational) consumer behaviour as conventionally assumed in microeconomic theory and therefore endogenous. For this reason the COLI claims to possess a theoretical (microeconomic) foundation of an index function of a CPI. It is assumed that households are engaged in utility maximisation subject to the restriction of a given total expenditure (or income) M and a given vector $\mathbf{p}$ of prices. The COLI allows for substitutions among the same goods in response to varying relative prices, while the COGI keeps everything constant except prices. Therefore the COLI will in general display lower inflation rates, which not infrequently may be welcomed (politically), and the difference $P_{0t}^{L} - P_{0t}^{K} > 0$ is called "substitution bias" (of P_{0t}^{L}).

Those who advocate the COLI use to emphasise that the COLI enjoys a theoretical justification or "theoretical underpinning" (in contrast to the COGI which "only" may be justified by representing a weighted average of price relatives and by the "principle of pure price comparison" according to which the index is reflective of price changes only). Note, however, that (1) only defines the COLI P_{0t}^{K} but it does not give any hints about how to compile this index in practice because the c-function (and the utility function from which it is derived) is not observable. Given that we can't know how the "amount" of "utility" is related to the quantity of goods consumed we have to find ways to make the notion of a "constant-utility-index" or COLI nonetheless "operational" or "measurable". There are in principle three ways proposed in order to accomplish this task

1. historically the first approach was to define upper and lower bounds for a COLI (it is for example well known that under fairly general conditions $P_{0t}^P \leq P_{0t}^K \leq P_{0t}^L$ holds),

2. then some attempts were undertaken in order to estimate (econometrically) "demand systems", that is systems of n demand equations for N goods from which the theoretical cost-functions as numerator and denominator of the COLI can be derived (and also the shape of the Engel-curves and the estimates of some parameters such as various "elasticities". Those may be interesting regarding the economic interpretation of the empirical results). However, as this approach turns out to be extraordinarily difficult to carry out in practice, it became more and more popular, to

3. make use of the theory of "superlative indices" developed by [12][13]W. Erwin Diewert (1976, 1978) according to which certain observable price indices[1] - each "using the quantities in the base period as well as in the current reference period as weights in a symmetric fashion" - are capable of approximating a COLI derived from a fairly general (or "flexible") demand function.

The first approach certainly is less promising from a practical point of view of compiling an official CPI on a monthly basis, because it can at best provide intervals only rather than an exact numerical value. The second and in particular the third approach may appear more pertinent and successful.

The focus of our paper therefore is on the second and third approach. For this purpose we have undertaken both, a theoretical analysis of the assumptions explicitly (or implicitly) underlying Diewert's theory of superlative indices, and an empirical study of demand systems (using data taken from

[1]Such as $P_{0t}^F = \sqrt{P_{0t}^L P_{0t}^P}$ (Fisher price index) or $P_{0t}^T = \prod(\frac{p_{it}}{p_{i0}})^{\bar{s}_i}$ (Törnqvist price index) where $\bar{s}_i = 1/2(s_{i0} + s_{it})$ is the arithmetic mean of the weights of the Laspeyres and Paasche price index formulas.

the official German Family Budget Surveys EVS). To our knowledge there are no studies of demand systems of a comparable broad scope (comparing various systems) to be found in the literature to date.

2 The superlative index approach: assumptions and problems

2.1 Setting the stage

Given the immense problems with the Demand Systems Approach (DSA) Diewert's Superlative Index Approach (SIA) was well accepted and hailed everywhere on the part of the price statisticians as it promised to have solved finally a seemingly insurmountable measurement problem. The SIA is generally seen as the first and possibly (to this day at least) only method that allows the unobservable COLI (or an approximation of it) to be compiled validly in practice using observable price and quantity vectors only. The message of the SIA is usually understood as follows: Diewert was able to show that certain well known price indices P were "exact" for a specific cost function (and correspondingly quantity indices Q exact for a specific utility function). This means that if preferences of a household are following a certain utility function (or cost function derived from it) the COLI takes the form of a certain index (like Fisher P^F, Törnqvist P^T or Walsh P^W), which then is said to be "exact" for this particular type of utility or cost function. If this function is "flexible" in a sense to be defined later, the corresponding index is called "superlative" by Diewert. In fact P^F, P^T and P^W are the most prominent representatives of this family of "superlative" price indices.

In this part we try to show that this remarkable result of the SIA has its price in that it requires some restrictive assumptions which are unlikely to be met in reality and that its relevance notoriously seems to be greatly exaggerated. Before going into detail of explicit and implicit assumptions made in developing the SIA it seems to be pertinent to state right at the outset how (we believe that) the message of "superlative indices" probably is understood by the ordinary price statistician. It seems to us that in his view the following five statements are true

1. There are only small number of index function proved to be superlative

as for example P^F, P^T or P^W (and the corresponding quantity indices Q^F, Q^T and Q^W of Fisher, Törnqvist and Walsh).

2. Hence to be superlative is a rare and most honourable distinction which most of the hitherto familiar and popular indices like Laspeyres P^L, Paasche P^P and so do not deserve.[2]

3. To be "superlative" means that it is distinctly shown that the index in question is (approximately) equal to the COLI and thus valid for any utility function whatsoever[3] (and any consumer if he behaves rationally) which dispenses us once and for all from studying behaviour consumer empirically .

4. It is in particular no longer relevant to study how the average household responds to changes of relative prices by substituting away from goods that became dearer. The so called "substitution bias" is correctly measured as difference between a non-superlative index and a superlative index, for example $P^L - P^F$.

5. Although the underlying COLI-Theory had been developed for the case of one single household only and the compilation of a price index in one stage only[4] the more realistic many-households and multiple-stage compilation case does not require fundamental changes of the SIA.

In what follows we attempt to show that this interpretation of Diewert's SIA is false in all five points and that the approach requires quite restrictive assumptions which are not likely to be met in real households' behaviour. Lacking realism of the theory's assumptions matters and ensues also that

[2]In a similar vein it is frequently argued that it is an advantage as such - and therefore justification - of an index design (like for example the method of chain indices) over other indices only because it comes closer to a superlative index than the other indices do.

[3]The notion of a "flexible" function is widely understood as approximating closely enough an "arbitrary" function (or simply "any" function one might think of) so that there is no point in dealing any more with specific functions.

[4][15]Diewert (2001) is one of the rare occasions where Diewert explicitly discussed problems (for a COLI) involved in the fact that statistical agencies compile (aggregate) price indices in two or more stages in practice (using various "component subindices" as sub-indices on various levels of aggregation). He showed that indices like PF and PT satisfy consistency in aggregation only approximately. He did not, however, refer to problems of separability of utility functions that is to the microeconomic foundation of the subindices as opposed to the overall index.

the theory's relevance is much more limited than what is usually believed. This is even more a point as the COLI claims precedence over all other index approaches and index formulas just because it possesses a theoretical "foundation" while other indices do not.[5]

A number of steps is needed in order to derive the result that a certain index function is superlative. In each step some assumptions are implied. We discuss assumptions in the sequence of these steps, beginning with some most fundamental assumptions of the COLI theory and proceeding in the way "superlativity" of distinct index function is proved.

To begin with (sec.2.2) it is assumed that households decide on their purchases by maximising their utility. For this it is necessary that the vectors q of quantities of goods and the as great as possible "amount" of "utility" u are related to one another somehow. This leads to the notion of a utility function $u = f(\mathbf{q})$. As $\mathbf{q}$ is related to u it is maintained that quantities are endogenous or "explained" in COLI framework and no longer assumed as given or "exogenous" as they are in other index theories. However, a utility function in turn requires some quite restrictive assumptions. We are going to show how these general requirements of a utility function may be inferred from "rational" consumer behaviour.

In doing so we are dealing so far only with the COLI in general. In order to move in a further step to the more specific SIA topic we have to consider specific functional forms for the utility function and related functions, like the "costs function". This will be done in sec.2.3 where we introduce two specific functional forms used by Diewert when he derived some families of superlative indices. There we also demonstrate how a functional form is related to its corresponding price and quantity index and what is implied when the form is called "flexible".

2.2 Preference formation and utility maximising (COLI assumptions)

It is generally accepted that the SIA needs to presuppose that households are engaged in utility maximisation. The COLI theory as basis of the SIA starts with a single household. Before a utility function exists that can reasonably

[5] For [27]Triplett (2001) it is clear that the microeconomic theory underlying the COLI is the only theory we have, and all other indices have no underlying conceptual framework.

be maximised, a household[6] must have preferences (predilections and dislikes) and a preference order of quantity vectors $\mathbf{q}$ (or combinations of quantities $\mathbf{q}$ of N commodities). Unlike a utility function $f(\mathbf{q})$ the more fundamental "preference order" only provides a ranking of commodity combinations in ascending order of desirability (preference) where not yet numerical values of the amount of utility u are assigned to vectors $\mathbf{q\prime} = [q_1 \ldots q_N]$ as it is the case in a utility function $u = f(\mathbf{q})$. In order to provide a consistent ranking the commodity combinations have to meet criteria that are anything but self evident, viz. completeness (the ranking ought to comprise all possible combinations), transitivity (permitting a one dimensional order), non-satiation[7] and continuity which in turn requires "homogeneous" goods (infinite divisibility where quality remains constant). As to preference formation it is assumed that a household feels desires and satisfaction independently of other households, so that all sorts of imitation and emulation behaviour (as often observed) are ruled out.

Moreover it is assumed that the household decides over quantities in a rational and consistent manner. Maximisation of utility (under budget constrains) presupposes

1. the ability and willingness on the part of consumers to collect the necessary informations about prices, availability in suitable outlets) and to make the required changes in the consumption in response to changing relative prices (reaction is immediate and well informed, ideally the information about the *given* supply is complete and always up to date);

2. the existence of a utility function (indifference surface) which in turn requires that certain mathematical conditions are met, and also

3. the existence of a budget constraint (isocost plane) where again some assumptions are implicitly made and finally it is important to note that the COLI model assumes, that

4. the utility function is constant over time and that households make (possibly infinitely small) changes in their consumption *solely in response to changing relative prices* and not, for example because of

[6] It only later turns to the more complex COLI theory of many-households which is of course closer related to the scope of a national CPI than the single household case.

[7] This means that for any combination of commodities there must exist at least one other combination which is preferred to it.

changes in tastes or constraints on the supply side[8] or as a consequence of other consumers' decisions.

Each of these assumptions is far from being a matter of course. It is in the first place not self-evident that households are engaged in utility maximisation. Also unlikely is the prevalence of a constant utility function over all goods and representative for all households serving as a precise description of consumer decisions in quantitative terms.

Ad 1: The assumption amounts to postulating an arguably idealised (extreme), infinitely flexible and active homo oeconomicus. The two Irish authors [25]Murphy and Garvey (2005) found out that contrary to their most plausible expectations, poor households were less adversely affected by inflation than the higher income groups. The received wisdom is that households who are better off have a greater range of choice among goods which are not only necessities so that they are supposed to be more flexible and more inclined to substitute goods that became relatively dearer. However, surprisingly their consumption pattern exhibited less flexibility than the one of less affluent consumers. The reason for this paradox seems to be that the rich may have been discouraged by the disutilities and (psychic) costs of search activities involved in constantly rationally rearranging ones consumption. They may be interested in avoiding inconvenience and they are in the comfortable situation that they can afford it. On the other hand the relatively poor may be forced to be more flexible due to their disadvantaged economic situation. Hence preferences and flexibility may in actual fact be a product of one's economic situation rather than exogenously determined. Not only preferences but also the propensity to actively rearrange consumption in response to price movement can be income or class determined.

Ad 2: In order to transform a preference order into a real valued utility function $u = f(\mathbf{q})$ according to which utility u depends only on the vector $\mathbf{q}$ of quantities of market purchased goods we have to assume a metric variable u and that marginal changes of quantities can be made that will sensibly affect u. This implies that the function $f(\mathbf{q})$ has got to be a continuous,

[8]For example a good is no longer available or offered only with some new features not wanted by the consumer. Ideally none of the phenomena that characterises a dynamic market should take place: goods disappear and new goods emerge on the market, quality changes etc. For all those reasons consumers may be forced to involuntarily change their demand induced by activities on the supply side of the market.

increasing (in elements of $\mathbf{q}$), quasi-concave[9] (to establish an unequivocal maximum), and twice continuously differentiable function over all parts of its domain. These assumptions are known as "regularity conditions" and they are reconcilable only with a limited number of functional forms so that not all imaginable functions are eligible for representing $f(\mathbf{q})$.

For the COLI theory it suffices to postulate such quite general properties of a utility function $f(\mathbf{q})$, however, for the SIA it is necessary to be more specific regarding the functional form of $f(\mathbf{q})$.[10] The problem with utility maximising is that we cannot be sure whether or not households in fact perform this optimisation exercise in their decision making.[11] The SIA rationale, however, decisively rests on the assumption that observed expenditures are resulting from utility maximisation. More specific it is assumed that the scalar product $\mathbf{p}_s'\mathbf{q}_s = \sum_i p_{is}q_{is}$ $(i = 1, \ldots, N$ and $s = 0, t)$ is equal to the value of the cost function[12] $c(\mathbf{p}_s, u_s)$ in both periods $s = 0$ and $s = t$. Interestingly from the mere *definition* of c as *minimum* costs (expenditure) follows that $\mathbf{p}_t'\mathbf{q}_0 \geq c(\mathbf{p}_t, u_0)$ and $\mathbf{p}_0'\mathbf{q}_t \geq c(\mathbf{p}_0, u_t)$ and therefore

$$P_{0t}^L = \frac{\mathbf{p}_t'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0} \geq \frac{c(\mathbf{p}_s, u_0)}{\mathbf{p}_0'\mathbf{q}_0} = \frac{c(\mathbf{p}_t, u_0)}{c(\mathbf{p}_0, u_0)} \tag{4}$$

and

$$P_{0t}^P = \frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_0'\mathbf{q}_t} \leq \frac{\mathbf{p}_t'\mathbf{q}_t}{c(\mathbf{p}_0, u_t)} = \frac{c(\mathbf{p}_t, u_t)}{c(\mathbf{p}_0, u_t)}. \tag{5}$$

In each equation the rightmost term is a COLI referring to a utility level of u_0 and u_t respectively.[13] It is common practice to refer not only to the same

[9] This is a curvature condition for $f(\mathbf{q})$. A concave utility function means somewhat simplified that average quantities of all goods are preferred to extreme or one-sided consumption (0% or 100% of the budget devoted to one specific good).

[10] We consider these additional assumptions in sec. 2.3.

[11] It is indeed an advantage the econometric demand-system-approach (DSA) enjoys over the SIA that empirical results (like ours) indicating a poor goodness of fit of the respective models to data may be taken as a hint that household possibly are not acting as utility maximiser so that the theoretic assumptions may not be realistic.

[12] That is the minimum costs (expenditure) of achieving a utility level u_s under a prevailing price vector ps (or to put it differently we chose that very vector $\mathbf{q}$ of quantities for which the expenditure $\mathbf{q'p}_s$ under the price regime of period s is a minimum). A unique value c of $c(\mathbf{p}_s, u_s)$ requires that the above mentioned conditions are met concerning the utility function $u_s = f(\mathbf{q}_s)$ from which the function $c(\mathbf{p}_s, u_s)$ is derived.

[13] This is in the literature sometimes called Laspeyres-Konüs and Paasche-Konüs (price)

utility function $f(\mathbf{q})$ but also to same value u of this function ($u = f(\mathbf{q})$) in both numerator and denominator of the COLI (price) index which thus is defined as equation (1) in the introduction. It is important to note that the assumption that households utility maximisation on the part of the households is crucial. Otherwise we could not equate $\mathbf{p}_s'\mathbf{q}_s = c(\mathbf{p}_s, u_s)$ and bring the observed indices P^L and P^P into play and relate them to the unobserved COLI. Once utility maximisation does not take place a superlative index will no longer approximate the COLI.

Ad 3: In order to define the budget constraint (or isocost plane) it is necessary to specify how total consumption expenditure[14] M is related to income[15] and the commodity prices $p_1, \ldots, p_N$ are generally assumed as being determined exogenously. Households are assumed to be price takers which means prices are given, independent of quantities purchased and the same for all households.[16] Also M is given which means that we do not explain a household's time allocation and labour supply on the basis of decisions over leisure time t_L and working hours t_W with reference to a utility function $f(\mathbf{q}, t)$ where $t = t_L + t_W$.

Along with the above mentioned regularity conditions concerning $f(\mathbf{q})$ such assumptions are needed in order to have a *linear* budget constraint otherwise we would run into difficulties establishing an optimum as a unique tangency point of the isocost plane with the indifference surface. Other necessary assumptions are

- Due to limited resources (finite income) an important constraint is also

index respectively. The assumption of "homothetic" preferences introduced later is valuable as a simplification in that it then does not matter to which period the utility level refers (so that Laspeyres-Konüs and Paasche-Konüs coincide).

[14] One may make a distinction between consumption (creating utility) and consumption expenditure. The difference is not only time but also household production as many purchases are subject to a significant amount of processing (e.g. cooking) within the households. Use of goods must also be distinguished from acquisition of goods. It is assumed that goods are acquired by purchases and not received as payments in kind, gifts or so.

[15] The familiar assumption her is that all income is spent for purchases. This rules out that households take care of future consumption by saving. We then have a more comfortable single period utility maximisation problem only.

[16] [14]Diewert (2000) considered at length a model in which he relaxed the assumption that "prices are constant across households". The resulting equations (relating a disaggregated Laspeyres or Paasche index to the "usual" Laspeyres or Paasche index) are quite complicated.

that budget shares of all N goods ought to add up to unity (100%).

- The (linear) budget constraint is a function (in M and $\mathbf{p}$) which should be homogenous of degree zero in M and $\mathbf{p}$ (to ensure that no "money illusion" prevails). This assumption is needed to make sure that only relative prices rather than absolute prices matter when households make rearrangements (substitutions) in their consumption.

Ad 4: It is important to note that these substitutions are made on a constant (invariant) utility function solely in response to changes in the relative prices (over which the household has no command). To increase or decrease one's demand should not result from changing tastes or reacting to activities on the supply side. Ideally higher or lower income should not influence the preferences order.[17] However, in reality tastes are constantly changing with the passage of time and they are also affected from ageing, technology and supply. The set of goods over which quantified preferences (that is utility values) are defined is varying, new goods emerge an old ones disappear from the market. In reality changes in consumption patterns are clearly *not attributable only to decisions on the demand side of the market.*

In summary all these assumptions are inherent in the COLI approach on the basis of which also both approaches, SIA and DSA are built. It is by no means less realistic to make the assumption of a constant (and observable) basket of goods than to assume a constant function $f(\mathbf{q})$ relating quantities to the unobservable "utility". Furthermore we have seen that to legitimately call the COLI approach theoretically superior to the COGI approach depends on the realism of the underlying theory of consumer behaviour. And this in turn is contingent on quite a few conditions which are not likely to be met in real situations.

2.3 Functional forms and their corresponding indices (the SIA assumptions)

We now come to additional assumptions needed in Diewert's SIA. The preceding section made clear that to be eligible as a utility function (and as the corresponding cost function), or more general as an "aggregator function" is not easy. To serve as a utility function $f(\mathbf{q})$ the function rather needs to

[17] This refers to the issue of "homothetic" preferences we will discuss shortly.

meet the quite restrictive "regularity conditions" at least in relevant parts of its domain. It therefore cannot be any arbitrary function.

Before embarking on a discussion of suitable functional forms for $f(\mathbf{q})$ it may be interesting to note that [5]Barnett and Choi (2008) called in question Diewert's way to concentrate only on functions that exist in an algebraically closed form. They contend that there are many functions that can "only be tabulated"[18] and evaluated by "using partial sums of series of (Taylor) expansions" his method "spans only a strict subset of the general class" (of superlative indices) in question.[19] They also showed that all indices of the class of log-change indices[20] (of which P^T is a member) are "superlative".

Note that even if we confine ourselves to those functions for which a closed and parameterised algebraic form exists we may find an infinite number of superlative index formulas of which only a small subset is known and capable of a plausible interpretation.

To develop a theory of superlative indices and make use of the SIA four steps are needed:

1. A functional form has to be found which complies with all assumptions needed to reflect rational consumer behaviour, and
2. the price and quantity index corresponding to ("exact" for) this functional form have to be derived, and
3. it has to be demonstrated that the functional form in question is "flexible" in the definition of Diewert, and finally
4. as there is an abundance of superlative indices it is desirable to have some guidance in making a choice among these formulas.

[18] Examples they referred to are trigonometric or hyperbolic functions.

[19] [6]Barnett et al. (2003) therefore also wondered why only a small number of index numbers (of a potentially infinite number) have so far been found and maintained that the search process of an index number which is exact for a given functional form or - the other way round - for a functional form which is exact for a given index function is not yet formalised ("No simple procedure has been found for either direction. For example the miniflex Laurent aggregator function, originated by [7]Barnett and Lee (1985) . . . , is known to be second order, but no one has succeeded in finding the index number, that can track it exactly").

[20] See [29]von der Lippe (2007), pp. 226 - 254. These indices are also called (e.g. in [5]Barnett and Choi (2008)) "Theil Sato Indices".

Ad 1: It is useful to consider homothetic[21] or linear homogeneous utility functions as an interesting and convenient sub-set of utility functions. A function $g(x_1, ..., x_n)$ is called homogeneous of degree r if

$$g(\lambda x_1, ..., \lambda x_n) = \lambda^r g(x_1, ..., x_n). \tag{6}$$

The analytic merit of linear ($r = 1$) homogeneity is that

$$c(\mathbf{p}, u) = uc(\mathbf{p}) = f(\mathbf{q})c(\mathbf{p}), \tag{7}$$

where $c(\mathbf{p}) = c(\mathbf{p}, 1)$ is the "unit cost function" (minimum cost to acquire a utility level of $u = 1$)[22] which provides an enormous simplification. Homotheticity implies that the COLI is independent of the utility level and therefore the same for all income classes. Moreover all goods have unitary income elasticities, Engel curves (consumed quantities $\mathbf{q}$ as function of income) are straight lines through the origin[23] and expenditure shares w_i $(i = 1, .., N)$ are unaffected by changes in income (they are - as desired in the COLI theory - solely reflective of changes in relative prices). However, there is "an overwhelming amount of empirical evidence contradicting homotheticity of preferences" ([4]Barnett (1983), p. 218), so that the assumption of homotheticity is generally regarded as highly unrealistic. On the other hand the assumption is extremely convenient. Just as the value index V_{0t} is decomposed in a price and quantity index, P_{0t} and Q_{0t} respectively

$$V_{0t} = \frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_0'\mathbf{q}_0} = P_{0t}Q_{0t}, \tag{8}$$

called "product test", or weak factor reversal test[24], and in a similar manner V_{0t} is usually factored in the economic index as follows

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[21] It is common to make a distinction between "homothetic" and "non-homothetic" utility functions. A utility function f is defined (by Shephard) as homothetic if it can be written as a monotonic transformation of a linearly homogeneous function. However for all practical purposes it is tolerable to use "homothetic" and "linear homogeneous" as synonyms (like [16]Diewert did, p.6, footnote 8; this seems to be justified, see also [21]Kats (1970)).

[22] Diewert has shown that $c(\mathbf{p})$ and $f(\mathbf{q})$ satisfy the same regularity conditions.

[23] In this respect "quasi-homothetic" preferences are more general in the fact that the Engel curves "need not be forced through the origin" ([4]Barnett (1983), p.217).

[24] The (strict) factor reversal test requires the same functional form for P and Q (interchanging prices p and quantities q in the price index P results in the quantity index Q and vice versa).

$$V_{0t} = \frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_0'\mathbf{q}_0} = \frac{c(\mathbf{p}_t, u_t)}{c(\mathbf{p}_0, u_0)}, \tag{9}$$

assuming utility maximisation in both periods and using the homotheticity assumption

$$\frac{u_t c(\mathbf{p}_t)}{u_0 c(\mathbf{p}_0)} = \frac{f(\mathbf{q}_t)}{f(\mathbf{q}_0)} \cdot \frac{c(\mathbf{p}_t)}{c(\mathbf{p}_0)}, \tag{10}$$

where the term (the ratio of utility levels) is said to be the "economic quantity index", while the second factor is called "economic price index" or simply COLI.

In his theory of superlative indices Diewert studied in particular only two functional forms for $f(\mathbf{q})$ and (derived therefrom) the cost function (we quote the latter only), viz.

1. the quadratic mean of order r utility and cost function

$$c_r(\mathbf{p}) = \left(\sum_i \sum_j b_{ij} p_{it}^{r/2} p_{jt}^{r/2} \right)^{1/r} = (\mathbf{p}'\mathbf{B}\mathbf{p})^{1/r} \tag{11}$$

where $\mathbf{p} = \left[p_1^{r/2} ... p_N^{r/2} \right]$ the coefficient b_{ij} of $\mathbf{B} = \mathbf{B}\prime$ in the quadratic form $\mathbf{p}\prime\mathbf{B}\mathbf{p}$ have to meet certain restrictions, and

2. the normalised quadratic cost function ([17]Diewert (2009), pp. 18 - 23)

$$c_{NQ}(\mathbf{p}) = \mathbf{p}\prime\mathbf{b} + \frac{\frac{1}{2}\mathbf{p}\prime\mathbf{A}\mathbf{p}}{\boldsymbol{\alpha}\prime\mathbf{p}} \tag{12}$$

where the scalar $\boldsymbol{\alpha}\prime\mathbf{p}$ performs a sort of normalisation and again coefficients a and b are appropriately restricted. Both models are fairly general. The first nests the

- "homogeneous quadratic" function ($r = 2$) for which Fisher's ideal index is exact,
- the (homogeneous) translog function (r $\rightarrow$ 0)

$$\ln c_0(\mathbf{p}) = \beta_0 + \sum_i \beta_i \ln p_i + \frac{1}{2} \sum_i \sum_j \beta_{ij} \ln p_i \ln p_j \tag{13}$$

for which the Törnqvist index is exact,

- the generalised linear utility function ($r = 1$) $f(\mathbf{q}) = \mathbf{a}/\mathbf{p}$ with the corresponding general Leontief unit cost function $c(\mathbf{p}) = \mathbf{b}/\mathbf{p}$ and the CES function with suitable parameter restrictions ($a_{ij} = b_{ij} = 0 \; \forall \; i \neq j$) so that $c(\mathbf{p}) = a_0 \left(\sum_i a_i p_i^r\right)^{1/r}$.

A non-homothetic variant of (13) is given by (*[17]*Diewert (2009), p. 24)

$$\begin{aligned} \ln c_0(\mathbf{p}, u) \quad = \quad & \beta_0 + \sum_i \beta_i \ln p_i + \frac{1}{2} \sum_i \sum_j \beta_{ij} \ln p_i \ln p_j \\ & + \gamma_0 \ln(u) + \sum_i \gamma_i \ln p_i \ln(u) + \frac{1}{2} \gamma_{00} (\ln(u))^2 . \end{aligned} \tag{14}$$

Obviously with vanishing γ coefficients (14) specialises to the homothetic cost function (13). There is not much in the ways of economic theory and interpretation that can be said about why one functional form should be preferred over another one.[25]

Ad. 2: With a given functional form the price and quantity index has to be found which is exact for it. For example *[17]*Diewert (2009) has shown that the function

$$P_{0t}^{r=2} = \frac{c(\mathbf{p}_t)}{c(\mathbf{p}_0)} = \frac{\mathbf{p}_t' \mathbf{B} \mathbf{p}_t}{\mathbf{p}_t' \mathbf{B} \mathbf{p}_t}, \tag{15}$$

where $\mathbf{p} = [p_1 ... p_N]$ coincides with Fisher's ideal price index $P_{0t}^{r=2} = P_{0t}^F$. Hence P^F is exact for the "homogeneous quadratic" functional form. Correspondingly the Fisher's quantity index Q^F is equal to

$$Q_{0t}^{r=2} = \frac{u_t}{u_0} = \frac{f(\mathbf{q}_t)}{f(\mathbf{q}_0)}.$$

In a similar manner Diewert has shown that the Törnqvist *price* index is exact for the translog cost function (13).[26] The general form of a price index exact for quadratic mean of order r functional form is given by

[25] [28]Turvey (1999), an ingrained opponent of the COLI approach said "Writers on this [i.e. COLI] theory express no views of which functional form is most realistic". More about purely formal criteria in making a choice among functional forms can be found in [22]Lau (1986).

[26] He did not consider the Törnqvist quantity index. Note that the Törnqvist index does not satisfy factor reversibility. The index function even fails the weaker product test.

$$P_{0t}^{r} = \left[\sum_i w_{i0}\left(\frac{p_{it}}{p_{i0}}\right)^{r/2}\right]^{1/r}\left[\sum_i w_{it}\left(\frac{p_{it}}{p_{it}}\right)^{-r/2}\right]^{-1/r}, \tag{16}$$

if $r \neq 0$ and w_{it} denoting expenditure shares of commodities $i = 1, \ldots, N$ in period $t = 0, t$. This constitutes a family of infinitely many superlative (price) indices for $-\infty < r < +\infty$. Accordingly if $r = 0$ we get

$$P_{0t}^{r=0} = \prod_i \left(\frac{p_{it}}{p_{i0}}\right)^{\frac{w_{i0}+w_{it}}{2}} = P_{0t}^{T}. \tag{17}$$

The corresponding superlative quantity indices are gained by simply substituting q_{it}/q_{i0} for p_{it}/p_{i0} It is important to note that there are an infinite number of superlative indices on the basis of the quadratic mean of order r. This functional form, however, requires homothetic preferences. Using the non-homothetic cost function (14) Diewert could demonstrate that the Törnqvist index is exact for the utility level $u^* = \sqrt{u_0 u_t}$. For other levels $\tilde{u} \neq u^*$ the (no less) superlative price index $c(\mathbf{p}_t, \tilde{u})/c(\mathbf{p}_0, \tilde{u})$ will differ from P^T. So the SIA involves occasionally some additional assumptions. They may be without a doubt plausible, however not at all cogent. In this case of non-homotheticity[27] it is an assumption about the absolute level (amount) of "utility" that had to be introduced. Another situation where such an assumption is involved is *[17]*Diewert's (2009) finding (pp. 21 -23) that

$$\text{a)} \quad P_{0t}^{A} = \frac{1}{2}\left(P_{0t}^{L} + P_{0t}^{P}\right) \tag{18}$$

or

$$\text{b)} \quad P_{0t}^{H} = \left[\frac{1}{2}\left(P_{0t}^{L}\right)^{-1} + \frac{1}{2}\left(P_{0t}^{P}\right)^{-1}\right]^{-1} \tag{19}$$

is exact for the normalised quadratic cost function (12) and therefore again "superlative". However, this applies only if we set

a) $\boldsymbol{\alpha} = \mathbf{q}_t$ and

b) $\boldsymbol{\alpha} = \mathbf{q}_0$ respectively.

---

[27] The result (of Diewert) concerning the index P^T of Törnqvist is to our knowledge the only proof of superlativity where the (notoriously unrealistic) assumption of homothetic preferences is being relaxed.

The corresponding quantity indices are a) harmonic mean Q^H_{0t} (corresponding to P^A_{0t})[28] and b) arithmetic mean Q^A_{0t}.

Note that the vector $\boldsymbol{\alpha}$ can also be specialised in a great number of other ways. We tried for example $\boldsymbol{\alpha} = (\mathbf{q}_t + \mathbf{q}_0)/2$ and got another superlative price index for which, however, we could not find an obvious (or economically luminous) interpretation, e.g. in terms of well known index formulas, like P^L, P^P or so.

It is also remarkable that an index function can be exact for different functional forms for the cost (or utility) function (and there possibly only for certain restrictions concerning the parameters) and conversely the same cost (or utility) function fit to different superlative index formulas. This may be seen as a challenge for an economic interpretation of the meaning of "superlative".

Diewert defined another infinitely large class of superlative indices by making use of the product test, defined in (8), showing that if P_{0t} is superlative this applies also to the corresponding "indirect" quantity index. Likewise if Q_{0t} is superlative the corresponding indirect price index $\tilde{P}_{0t} = V_{0t}/Q_{0t}$ is also superlative.[29] Direct and indirect indices coincide whenever the index function is factor reversible which is the case with Fisher's index.

In this way (via considering indirect indices) also the Walsh index formula received the honour of generating superlative indices. The quadratic mean of order $r = 1$ superlative price index

$$\begin{aligned} P^{r=1}_{0t} &= \left[\sum_i w_{i0}\left(\frac{p_{it}}{p_{i0}}\right)^{1/2}\right] \cdot \left[\sum_i w_{it}\left(\frac{p_{it}}{p_{it}}\right)^{-1/2}\right]^{-1} \\ &= V_{0t}/\frac{\sum q_{it}\sqrt{p_{i0}p_{it}}}{\sum q_{i0}\sqrt{p_{i0}p_{it}}} = V_{0t}/Q^W_{0t}, \end{aligned} \tag{20}$$

turns out to be the indirect price index of Walsh $\tilde{P}^W_{0t} = P^{r=1}_{0t}$ this index is also superlative. Note that the indirect Walsh price index differs from the direct Walsh price index

[28] This index is said to originate from Drobisch in Germany, however Diewert refers to Bowley and Sidgwick.

[29] Interestingly Diewert did not make use of such considerations in terms of indirect indices in a consistent manner. One may for example wonder why the indirect quantity index of Törnqvist V/P^T is not "superlative".

$$P_{0t}^{W} = \frac{\sum p_{it}\sqrt{q_{i0}q_{it}}}{\sum p_{i0}\sqrt{q_{i0}q_{it}}} \neq \tilde{P}_{0t}^{W} = P_{0t}^{r=1}. \tag{21}$$

So strictly speaking it is the indirect, rather than the direct Walsh index which is superlative (a quadratic mean of order $r = 1$ index). In a similar vein the quadratic mean of order $r = 1$ superlative quantity index is given by

$$Q_{0t}^{r} = \left[\sum_i w_{i0}\left(\frac{q_{it}}{q_{i0}}\right)^{r/2}\right]^{1/r}\left[\sum_i w_{it}\left(\frac{q_{it}}{q_{it}}\right)^{-r/2}\right]^{-1/r}, \tag{22}$$

so that for$r = 1$ we get $\tilde{Q}_{0t}^{W} = V_{0t}/P_{0t}^{W} = Q_{0t}^{r=1}$ saying that the indirect Walsh quantity index is again a superlative index (now a quantity index). For pairs of indices satisfying the product test there is no difference between a direct and an indirect index. Also $V_{0t} = Q_{0t}^{H}P_{0t}^{A} = P_{0t}^{H}Q_{0t}^{A}$ so that all four indices are superlative (be it directly or indirectly).

Ad. 3: For a price (quantity) index P (Q) to be superlative requires not only that P is exact for $c(\mathbf{p})$ and Q for $f(\mathbf{q})$ but also that $f(\mathbf{q})$, and $c(\mathbf{p})$ respectively is a flexible functional form. What is meant by "flexible". According to Diewert a twice continuously differentiable function $f(\mathbf{q})$ is flexible if it provides a second order approximation to another function $f^{*}(\mathbf{q})$ around the point $\mathbf{q}^{*}$, meaning that "the level and all of first and second order partial derivatives of the two functions coincide at $\mathbf{q}^{*}$."[30] More specific one may write (in analogy to [5]Barnett and Choi (2008), p.4)[31]:

$$f(\mathbf{q}^{*}) = f^{*}(\mathbf{q}), \frac{\partial f}{\partial \mathbf{q}}|_{\mathbf{q}=\mathbf{q}^{*}} = \frac{\partial f^{*}}{\partial \mathbf{q}}|_{\mathbf{q}=\mathbf{q}^{*}} \text{ and } \frac{\partial^{2} f}{\partial \mathbf{q}\partial \mathbf{q}'}|_{\mathbf{q}=\mathbf{q}^{*}} = \frac{\partial^{2} f^{*}}{\partial \mathbf{q}\partial \mathbf{q}'}|_{\mathbf{q}=\mathbf{q}^{*}} \tag{23}$$

Instead of "another" function f^{*} we also find the notion of an "arbitrary

[30] [16]Diewert (2008), p. 52. [17]Diewert (2009), p. 13, also stresses the requirement that $\mathbf{q}$ (and $\mathbf{q*}$) need to belong to the "regularity region" (of the utility function). What is said about $f(\mathbf{q})$ and $\mathbf{q*}$ of course also applies mutatis mutandis to the function $c(\mathbf{p})$ around the point $\mathbf{p*}$. See also [3]Barnett (1983b)for different definitions of "second order approximation".

[31] For a graphical interpretation see also [29]von der Lippe (2007), p. 109. It is beyond the scope of our paper to demonstrate exemplary what has to be done in order to show that a specific functional form in fact is "flexible" in the definition of Diewert.

linear homogenous function".[32] It is important to note that the approximation is evaluated at the point $\mathbf{q^*}$ of the two function in question, so that the situation may well differ substantially at other points. Furthermore flexibility is not the only desirable property of a functional form. There are other equally important criteria which may not be reconcilable with flexibility.[33]

Hence flexibility which makes an exact index superlative seems to be a purely formal criterion for which a "luminous" economic interpretation is lacking. Moreover the criterion should also be set against other more restrictive and unrealistic assumptions made in the course of proving the superlativity of an index formula. One may for example argue, that homothetic preferences are assumed in most of the SIA results (for example concerning Fisher's index P^F) such that the income level is irrelevant for the choice of a vector $\mathbf{q}_t$ as opposed to $\mathbf{q}_0$ which is undisputedly unrealistic.[34] This implies to focus exclusively on the substitution bias (of a non-superlative index as by contrast to a superlative one) while in the case of non-homothetic preferences the combined income and substitution bias may well offset the substitution bias.[35] So to be adequate for non-homothetic preferences may be more valu-

[32] [16]Diewert (2008), p. 15. [22]Lau (1986), p. 1539) emphasised explicitly "the ability ... to approximate arbitrary but theoretically consistent behaviour through an appropriate choice of the parameters". Aside from sounding a bit contradictory (theoretical consistency or regularity is anything but "arbitrary") this definition of "flexibility" underscores the idea of having left a sufficient number of free parameters to reflect different decisions of consumers and to generate for example different values for certain elasticities, expenditure shares, rates of substitution etc. Accordingly he wrote "so that ... their own and cross price elasticities are capable of assuming arbitrary values ... subject only to the requirements of theoretical consistency" (p. 1540). Arguing that the normalised quadratic function is flexible Diewert also pointed out that it has the "minimal number of free parameters that is required ... to be flexible."([17]Diewert (2009), p. 20)

[33] [22]Lau (1986) lists five criteria. These are in addition to flexibility theoretical consistency, a (wide) domain of applicability (whether consistency is globally, over the whole domain, or only locally valid), computational facility and factual conformity (according to which he rejects for example all those functional forms which generate linear Engel curves, as this appears to him highly unrealistic).

[34] Constant returns to scale is the production theory counterpart to homotheticity in consumption theory.

[35] This is in principle an argument of [18]Dumagan and Mount (1997). Their theoretical and empirical work amounts in no small measure to a vindication of the Laspeyres formula. They also criticise from the point of view of microeconomic theory the usage of both weights $\mathbf{q_0}$ as well as $\mathbf{q_t}$ (or expenditure shares w_{i0} as well as w_{it}) in a symmetrical fashion which interestingly - at least to our knowledge - takes place in all superlative index formulas. There is no superlative index which refers to the weights of one of the two periods only.

able than to be able to provide a second order approximation (rather than a first order only as is that case with P^L).

Ad. 4: As there are quite a few equally "superlative" indices it seems desirable to make an "informed" choice among these index formulas. In this respect it is often believed that all superlative indices approximate each other sufficiently close because Diewert has shown that "all the superlative index formulae ... approximate each other to the second order around any point where the two price vectors ... are equal and where the two quantity vectors ... are equal" ([14]Diewert (2000), p. 61). This result is generally understood as justifying the view that it does not matter much which superlative index is taken as they will display a similar order of magnitude for whichever data. The problem, however, is that the "equal (or proportional) price and quantity point" ([14]Diewert (2000), p.65) where $\mathbf{p}_t = \lambda \mathbf{p}_0$ and $\mathbf{q}_t = \lambda \mathbf{q}_0$ respectively[36] is relatively uninteresting. This is so because in that case all reasonable indices will be identically λ anyway. Moreover [20]Hill (2006) found out that the approximation theorem referring to the quadratic mean of order r (which is a flexible functional form for all values of r), does not apply to extreme values of r. It works well in the range $0 \leq r \leq 2$ but it can yield with a skewed distribution of price relatives and $|r| \to \infty$ differences between P^r and P^{-r}which may be much greater than the difference $P^L - P^P$. So Hill concluded that "the economic approach does not by itself solve the index number problem, since it does not tell us which superlative index should be used".

In summary we may conclude that the COLI-theory in general and the SIA (built on this theoretical foundation) in particular need a number of quite restrictive assumptions so that in "the perennial question of the realism of the theory's assumptions" ([27]Triplett (2001), p. 318) *in our view* the COLI may well come out at the losing end compared to the COGI. There are also other aspects we could not deal with here, that may cast doubt on the general belief of COLI advocates, that the Laspeyres index falls far behind the Fisher index for example. The superlative quantity index for example plays a somewhat dubious role. When $Q_{0t}^F > 1$ is superlative this means that the standard of living (level of utility) increased because Q_{0t}^F approximates $u_t/u_0 = f(\mathbf{q}_t)/f(\mathbf{q}_0)$. On the other hand P_{0t}^F measures the minimum cost of maintaining a constant level of utility.[37] The relevance (representativity) of it

[36] The "equal" point is of course the case $\lambda = 1$.

[37] It is the word "minimum" which Triplett seems to have forgotten when he levies the

might be questioned when $u_t > u_0$ just like is being criticised for its constant basket which is constantly becoming unrepresentative and irrelevant with the passage of time. Other vexing problems with the COLI theory are problems involved in

- proceeding from the single household case to the multiple household case[38], and
- in justifying the usual multistage index compilation of official statistics, not only in terms of aggregation properties of index functions[39] in terms of sub-utility maximisation as it would be required in the COLI framework.

However this would go beyond the topic of this paper.

3 Demand Systems

After having analysed Diewert's Superlative Index Approach, we will now turn to the Demand Systems Approach to approximate the COLI. There is more than one demand system in accordance with the usual assumptions of the utility maximising household. Hence a choice has to be made among a number of functional forms, each of which renders specific functions for the indirect utility function or the cost function and above all for the budget shares. This choice may be oriented at various desirable properties of the respective functional forms. The specific regression functions may for example comply with the (microeconomic) "regularity requirements" of utility and demand functions only within a more or less wide range of possible values for the parameters of the function rather than "globally". In addition

following argument against the Laspeyres price index P^L ([27]Triplett (2001), p. 324):Any index different from the COLI means that the utility level is not constant such that "stabilising the Laspeyres" by the monetary authorities "implies diminishing living standard". However P^L traces the cost of buying the same basket of goods which of course involves the same level u given a constant utility function. It is only the cost $c(\mathbf{p}_t, u_0)$ which will *by definition* be smaller than $\mathbf{p_0}'\mathbf{q_t}$.

[38] The question which of the two weighting schemes (plutocratic or democratic) is preferable is still not resolved.

[39] For example Diewert discussed this type of considerations only in [14]Diewert (2000), p. 63ff.

to this criterion of a sufficiently wide "domain of applicability" there are some other properties that ought to be considered, such as "flexibility"[40] or "computational facility" and "factual conformity"[41]. As regards the criteria for functional forms mentioned above the Almost Ideal Demand System (abbreviated by the nowadays unfortunate acronym AIDS) of Deaton and Muellbauer (1980) and the more general system QUAIDS (quadratic AIDS) of [2]Banks et al. (1997) represent a fair compromise.

3.1 Almost Ideal Demand System

The starting-point of [11]Deaton and Muellbauers (1980) AIDS is a cost function of the price independent generalized logarithmic (PIGLOG) class of preferences:

$$\ln c\left(\mathbf{p},u\right)=\left(1-u\right)\ln a\left(\mathbf{p}\right)+u\ln b\left(\mathbf{p}\right), \tag{24}$$

where the utility level u generally lies between *0* (subsistence level) and *1* (bliss point) and where

$$\ln a\left(\mathbf{p}\right)=\alpha_0+\sum_{i=1}^{N}\alpha_i\ln p_i+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}^{*}\ln p_i\ln p_j \tag{25}$$

and

$$\ln b\left(\mathbf{p}\right)=\ln a\left(\mathbf{p}\right)+\beta_0\prod_{i=1}^{N}p_i^{\beta_i}. \tag{26}$$

Filling in equations (25) and (26) in (24) results in the AIDS cost function

[40] Diewert defines flexibility in his Lecture Notes (ch. 4) as follows: it is a function f(q) that "can provide a second order approximation to an arbitrary function f* around any (strictly positive) point q* in the class of the linearly homogenous functions." And by second order approximation is meant: "A twice differentiable function f(q) ... can provide a second order approximation to another such function f*(q) around the point q* if the level and all of the first and second order partial derivatives of the two functions coincide at q*."

[41] According to [22]Lau (1986) this is not met if for example the system can only yield linear Engel-curves (which e.g. applies to AIDS as opposed to QUAIDS). Also Lau had shown that it is not possible to reconcile the above mentioned criteria so that a compromise is called for.

$$\ln c\left(\mathbf{p},u\right) = \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}^{*} \ln p_i \ln p_j + u\beta_0 \prod_{i=1}^{N} p_i^{\beta_i}, \tag{27}$$

which is homogeneous of degree one in prices $\mathbf{p}$ if the following parameter restrictions hold:

$$\sum_{i=1}^{N} \alpha_i = 1, \quad \sum_{i=1}^{N} \beta_i = 0, \tag{28}$$

$$\sum_{i=1}^{N} \gamma_{ij}^{*} = 0 \quad \forall j, \quad \sum_{j=1}^{N} \gamma_{ij}^{*} = 0 \quad \forall i, \tag{29}$$

$$\gamma_{ij}^{*} = \gamma_{ji}^{*}. \tag{30}$$

By multiplying both sides of Shephard's Lemma $\partial c\left(\mathbf{p},u\right)/\partial p_i = q_i$ with $p_i/c\left(\mathbf{p},u\right)$ we get

$$\frac{\partial \ln c\left(\mathbf{p},u\right)}{\ln p_i} = \frac{p_i q_i}{c\left(\mathbf{p},u\right)} = w_i, \tag{31}$$

so that we can derive a budget share equation from (31):

$$w_i = \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j + \beta_i u \beta_0 \prod_{i=1}^{N} p_i^{\beta_i}. \tag{32}$$

As for a utilitiy maximizing household $c\left(\mathbf{p},u\right) = M$, we can derive the utiltiy level u via the indirect utility function:

$$v\left(\mathbf{p},M\right) = \frac{\ln c\left(\mathbf{p},u\right) - \ln a\left(\mathbf{p}\right)}{\ln b\left(\mathbf{p}\right) - \ln a\left(\mathbf{p}\right)} = u. \tag{33}$$

Putting the utiltiy level u from equation (33) in (32), we get the common AIDS budget share equation:

$$
\begin{aligned}
w_i &= \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j + \beta_i \left(\frac{\ln c(\mathbf{p},u) - \ln a(\mathbf{p})}{\ln b(\mathbf{p}) - \ln a(\mathbf{p})} \right) \beta_0 \prod_{i=1}^{N} p_i^{\beta_i} \\
w_i &= \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j + \beta_i \left(\ln M - \ln a(\mathbf{p}) \right) \\
w_i &= \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j + \beta_i \ln \left(\frac{M}{P} \right),
\end{aligned} \tag{34}
$$

where P is a price aggregator

$$
\ln P = \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i \ln p_j \tag{35}
$$

and where

$$
\gamma_{ij} = \frac{1}{2} \left(\gamma_{ij}^* + \gamma_{ji}^* \right). \tag{36}
$$

The parameter restrictions (28) - (30) of the cost function (27) should in general also hold for the budget share equation (34). Only the restrictions (29) and (30) have to be changed slightly due to equation (36) to:

$$
\sum_{i=1}^{N} \gamma_{ij} = 0 \quad \forall j, \tag{37}
$$

and

$$
\gamma_{ij} = \gamma_{ji}. \tag{38}
$$

As [11]Deaton and Muellbauer (1980) show, adding-up ($\sum w_i = 1$) requires the parameter restrictions (28) und (37) to be satisfied, the restriction (37) ensures the homogeneity of degree 0 in $\mathbf{p}$ und M of the budget share equation (34) and restriction (38) is the requirement for a symmetric Slutsky matrix. The AIDS can either be estimated by directly imposing the restrictions (28) - (30) or by testing the restrictions after having estimated the model without restricitions. Taking the parameter restricitons (28), (37) and

(38) into account, the number of parameters to estimate of the AIDS budget share equation (34) can be determined using equation $N\,(N-1)\,/2+2N-2$.

[24]Lewbel (1991) extends [19]Gorman's (1981) famous Engel curve rank definition and defines the rank of a demand system to be the maximum dimension of the function space spanned by the engel curves of the demand system. Following this definition, [23]Lewbel (1987) shows that the AIDS is a rank two demand system which implies that it has linear Engel curves (not necessarily through the origin).

3.2 Quadratic Almost Ideal Demand System (QUAIDS)

Several studies, like for example Lewbel (1991), Blundell et al. (1993) and Banks et al. (1997), found - at least for some goods - empirical evidence for non-linear Engel curves. So Banks et al. (1997) extended the AIDS model to allow budget shares being quadratic in expenditure. The underlying indirect utility function of this rank three Quadratic Almost Ideal Demand System (QUAIDS) in its general form

$$v\,(\mathbf{p},M)=\left[\left(\frac{\ln M-\ln a(p)}{b(p)}\right)^{-1}+\lambda\,(p)\right]^{-1}, \tag{39}$$

is equivalent to a general indirect utility function of a PIGLOG demand system supplemented by the homogeneous of degree *0* in p function $\lambda\,(p)$. Using the already known form 25 from the AIDS for $\ln a(p)$ together with $b(p)=\prod p_i^{\beta_i}$ and $\lambda\,(p)=\sum \lambda_i \ln p_i$ we get from equation 39 to the specific indirect utility function of the QUAIDS

$$v\,(\mathbf{p},M)=\left[\left(\frac{\ln M-\alpha_0-\sum_k \alpha_k-\frac{1}{2}\sum_k\sum_j \gamma^*_{kj}\ln p_k \ln p_j}{\prod_i p_i^{\beta_i}}\right)^{-1}+\sum_i \lambda_i \ln p_i\right]^{-1}, \tag{40}$$

and to the cost function

$$\ln c\,(\mathbf{p},u)=\alpha_0+\sum_k \alpha_k \ln p_k+\frac{1}{2}\sum_k\sum_j \gamma^*_{kj}\ln p_k \ln p_j+\frac{\ln u \prod_i p_i^{\beta_i}}{1-\ln u\sum_i \lambda_i \ln p_i}. \tag{41}$$

By applying Shephard's Lemma on (41), we receive the QUAIDS budget share equations

$$w_i = \alpha_i + \sum_j \gamma_{ij} \ln p_j + \beta_i \ln \left(\frac{M}{P} \right) + \frac{\lambda_i}{\prod_k p_k^{\beta_k}} \left(\ln \left(\frac{M}{P} \right) \right)^2, \tag{42}$$

where we are using P of the form (35) instead of $a(p)$ to ease the comparison between AIDS and QUAIDS. To be consistent with the neoclassical assumptions of consumer theory, the QUAIDS has to fulfill the following parameter restrictions:

$$\sum_i \alpha_i = 1, \ \sum_{i=1} \beta_i = 0, \ \sum_{i=1} \lambda_i = 0, \ \sum_i \gamma_{ij} = 0 \ \forall j, \tag{43}$$

$$\sum_j \gamma_{ij} = 0 \ \forall i, \tag{44}$$

$$\gamma_{ij} = \gamma_{ji}. \tag{45}$$

The restrictions of (43) ensures adding-up, the restriction (44) is required for homogeneity of degree *0* in p and M of equation (42) and restriction (45) is a necessary condition for a symmetric Slutsky matrix. Taking the parameter restricitons (43), (44) and (45) into account, the number of parameters to estimate of the QUAIDS budget share equation (42) can be determined using equation $N(N-1)/2 + 3N - 3$ (number of parameters of the AIDS plus the $N-1$ additional λ_i parameters of the QUAIDS). Compared to the AIDS, the expenditure and price elasticities of the QUAIDS can vary with the level of total expenditure M, indicating the higher flexibilitxy of the QUAIDS. In the QUAIDS it is for example possible, that the same good is identified as luxury good for a household with low total expenditure and as a necessity good. A drawback of using the QUAIDS is the danger of obtaining fitted budget shares for high total expenditure values that fall outside the $[0, 1]$ domain of definition.

By having a look on the budget share equations (34) and (42) of the AIDS and QUAIDS, it is obvious, that the AIDS is nested in the QUAIDS (AIDS is the special case of the QUAIDS if $\lambda_i = 0$ for all i). This is a big advantage because we can easily compare the goodness of fit of the two models concerning our observed household data described in the next section.

4 Data

The empirical analysis is based on the sample survey of household income and expenditure, abbreviated by its German initials EVS (Einkommens- und Verbrauchsstichprobe).[42] The EVS is part of the German system of household budget surveys, which consists additional to the EVS of the annually conducted current household budget survey (Laufenden Wirtschaftsrechnungen, LWR). The EVS is a cross-section household survey, conducted every five years that started in 1962/63 in West Germany and was extended since 1993 to East Germany. As statistical unit a household is defined as a group of persons whose command over income is shared, independent of their kin relationship. By a net sample (fully completed questionnaires) of 55,110 voluntarily participating households in 2008, the EVS is the largest survey of its kind in the European Union. The EVS is a quota and not a random sample so that statistical errors cannot be calculated in the theoretically exact way. The annual current population survey of Germany, the Microcensus, serves as a benchmark for recruiting the participants, who earn a small honorarium. Some of the quotas cannot be achieved completely, so that a final weighting has to be conducted. To publish results on the federal state level, the EVS sample is weighted according to the criteria "type of household", "social position of the household's reference person" and "income bracket" for each federal state using the current Microcensus results.

The EVS consists of four parts:

1. Initial household interview, in which the socio-demographic information about every household member, the overall housing situation and the endowment with durable consumer goods are enquired.

2. An appendix to the initial household interview evaluating the financial assets and debts of the household.

3. A household book, containing all kinds of income - including public and in-kind transfers - and expenditures on all categories of private consumption, has to be kept for three month.

4. A sub-sample of 20% of the participating households has to keep for one month another household book (detailed log book) in which all expen-

[42] For a comprehensive description of the EVS see [26]Statistisches Bundesamt (2005).

ditures and purchased quantities on food, alcoholic and non-alcoholic drinks and tobacco products have to be noted in absolute detail.

For our purposes the detailed log book is of great value. By dividing the expenditures by the quantities, we know the individual prices at which the household purchased the commodities.

So we have a price variation in our cross section that enables us to estimate demand systems with data from only one year. In cases where the individual price data is not available, an econometric estimation of a demand system would only be possible by using time series of aggregated cross section data. The time series approach is only feasible if an income and expenditure survey is conducted every year, as it is the case in the US and the UK. The only yearly survey available in Germany is the already mentioned LWR, which is less detailed than the EVS and which is explicitly not recommended for any time-series analysis.

Taking into account the above described data limitations, the demand systems are estimated for nine staple foods listed in the detailed log book of the EVS: Milk, cream, eggs, butter, margarine, apples, bananas, mineral water and coffee. The choice of the specific commodities was driven by several considerations. First, the level of commodity detail available in the detailed log is very high. Second, the staple foods chosen are purchased by the majority of the

households, so that a sufficiently large sample size is achieved without making assumptions about the households with zero consumption of an item. Third, the staple foods chosen are homogenous goods, without considerable quality changes, new types appearing or wide differences in product characteristics. Fourth, food is purchased frequently which is not the case for durables. Fifth, the food consumption is assumed to be separable from the non-food consumption in consumers utility function, so that the cross price reactions between food and non-food commodity groups should be rather small. And finally the availability of absolute price data from the consumer price statistic is also better for the food-, beverage- and tobacco-group than for other commodity groups. To ensure the homogeneity of the consumers analyzed, the socio-demographic variables of the EVS are used to include only households in the sample that are consisting of couples with one or more children younger than 18 years old.

For research purposes the research data centre of the GFSO provides so-called Scientific-use-files containing household level micro data from the EVS.

The micro-data is made anonymous by only making a 98% random sample of all household files available and by a limitation of the number of variables for which data is provided.[43] The GFSO provided us with the Scientific-use-files of the EVS for 1988, 1993, 1998 and 2003. To calculate the COLI after estimation of the demand systems, we use freely available price data out of the GFSO consumer price statistics.

5 Estimation

The simplest way to estimate the parameters of a system of equations is to estimate each equation separately by ordinary least square (OLS) techniques. Here, this is not possible because of two reasons: First, we have cross-equation parameter restrictions in The AIDS and QUAIDS, which can only be tested if the equations were estimated simultaneously. Second, we have to worry about contemporaneous correlation of the error terms that would violate the OLS assumptions. Contemporaneous correlation of the error terms occurs if the error terms of the N-budget share equations of one household were correlated with each other. The probability for the occurrence of contemporaneous correlation is in our context very high. The error terms of each of our budget share equations contains influences such as demographic composition or income of the household on the budget shares which are not yet included in the model by exogenous variables. As these influences typically vary between households but not within the specific budget share equations of one household, contemporaneous correlation is likely to occur. Nevertheless we conduct a Lagrange-Multiplier (LM) test, to test our data on contemporaneous correlation. The null hypothesis of freedom from contemporaneous correlation has to be rejected for all data sets under consideration. So we have to choose an estimation procedure that accounts for the effect of contemporaneous correlation and ensures the imposition of cross-equation parameter restrictions. The nonlinearity of the AIDS and QUAIDS additionally constrain the potential set of estimators. Following the well known seemingly unrelated regression approach of [30]Zellner (1962), we use a feasible generalized nonlinear least square estimator (FGNLS) proposed by [9]Cameron and Trivedi (2005).

[43] We are very grateful to the GFSO for the provision of the Scientific-use-files and many useful advices concerning the data set.

As described in the previous section, we have four cross-section data sets (EVS data from 1988, 1993, 1998 and 2003) available for our AIDS and QUAIDS estimation. For lack of space we can't display all estimation and test results of the four data sets. Our focus will lay on the 1988 results, because they enable us to calculate long COLI- time-series. A selection of output tables can be found in the appendix of this paper, further results are available upon request. After having estimated the AIDS and QUAIDS budget share equations (34) and (42) using the EVS data set from 1988, 1993, 1998 and 2003, we first want to test the symmetry and homogeneity restrictions. As the fulfillment of the symmetry restriction ensures also the fulfillment of the homogeneity restriction because otherwise the adding-up condition would be violated, we have to estimate the AIDS and the QUAIDS once with only the homogeneity restrictions imposed (model 2) and once completely without any restrictions (model 3) imposed. Now we can calculate the Likelihood-Ratio (LR) test statistic to compare the fit of the unrestricted model (model 2 or model 3) and the restricted model (the primary model specification with symmetry and homogeneity restrictions imposed, called model 1) which is a special case of the other. As table 4 and table 5 show, the null hypotheses (the restricted model has the same goodness of fit as the unrestricted model) can be rejected for the AIDS and the QUAIDS for all years (except for the 1993 QUAIDS comparison of model 1 and model 2) with a level of significance smaller than 0.05. This means that the observed household data sets are not allowing the conclusion that the neoclassical assumptions of demand functions that are homogeneous of degree 0 in prices and expenditure and symmetric Slutsky matrices were fulfilled. So the behaviour of the households is not free from money illusion and for the Hicks cross-price elasticities the relation $\partial q_i^H/\partial p_j = \partial q_j^H/\partial p_i$ is not valid.

To complete the test of the theoretical restrictions of the AIDS and QUAIDS, we have to test the monotonicity condition, which requires a cost function that is monotonically increasing in prices and the concavity condition that the cost function is concave. [10]Chalfant et al. (1991) show that the monotonicity condition is fulfilled if the fitted budget shares of all N-goods of the model are laying in the interval between 0 and 1. Testing the concavity condition is by far more complex. A cost function is concave at a certain data point, if the matrix of the second-order derivatives of the cost function – the so called Slutsky matrix – is negative semi-definite. Most of the demand studies proof the negative semidefiniteness of the Slustky matrix by only checking the sign of the diagonal elements of the Slustky matrix.

As [1]Alley et al. (1992) show, the non-positivity of the diagonal elements of the Slustky matrix is only a necessary but not a sufficient condition for the negative semidefiniteness of the Slustky matrix. A necessary and sufficient condition for the negative semidefiniteness of the Slustky matrix is the non-positiviy of all the eigenvalues of the matrix. An additional drawback of many demand studies is that the proof of negative semidefiniteness of the Slustky matrix is only conducted at one data point, typically at the mean of the price and expenditure observations. By contrast, we are calculating the eigenvalues of the Slutsky matrix at each data point of our sample. Table 6 provides the percentage of observations which does not violate the monotonicity and concavity conditions. The above described test of the concavity condition is carried out both with the observed and the fitted values of the budget shares w_i. On the first view, the results of the concavity test display a broad rejection of the concavity condition. A closer inspection of the particular eigenvalues reveals that for the wide majority of the observations only one of the N-eigenvalues is not negative. So we can conclude similarly to [1]Alley et al. (1992) that our cost functions have a "weakly" non-concave shape. The rejection of the symmetry, homogeneity and concavity conditions for most of the estimated models is a clear sign that the households didn't behave in the neoclassical way. This finding is in line with most of the empirical demand analysis (as summarized for example by Cozzarin and Gilmour (1998)) and was already expected by [11]Deaton and Muellbauer (1980) when they formalized the AIDS. As already mentioned in the third section, the AIDS is nested by the QUAIDS. So we can conduct a LR test to see if the restricted model (AIDS) has the same goodness of fit as the unrestricted model (QUAIDS). Table 7 shows that for all four data sets the null hypotheses, that the AIDS has the same goodness of fit as the QUAIDS, can be rejected on a level of significance smaller than 0.05.

With the estimated parameters of the AIDS and QUAIDS for 1988 we calculate COLI time series from 1988 to 2009. The expenditure of the representative household to attain the cost of living can be calculated by inserting the price data for the nine commodities out of the official German consumer price statistics from 1988 to 2009. For the same nine goods, a traditional Laspeyres price index is calculated by using a weighting scheme obtained from the 2581 household observations of 1988s EVS that we have used to estimate the AIDS and QUAIDS. Table 8 presents the results of the AIDS- and QUAIDS-COLI and the Laspeyres type CPI with base year 1988. To compare our AIDS- and QUAIDS COLI with superlative index numbers, we calculate Fisher- and Törnqvist- price indices. As we need actual weighting scheme informations for the Fisher- and Törnqvist- price index calculation, we can only calculate index numbers for the three years with EVS data available. Table 8 and table 9 show, that not only the Laspeyres price index is exceeding the two COLIs but also the Fisher- and Törnqvist- price index are lying far above the two COLIs. These results are a strong drawback for the supperlative index approach.

6 Conclusion

The result of the empirical part was that the demand systems estimated had a poor goodness of fit (table 3) which may possibly indicate the consumption behaviour of households is not well explained by the usual assumptions of utility maximisation in microeconomics as they are materialised in the demand systems. In particular the demand-system and the superlative-index approach obviously don't fit together satisfactorily. The Fisher- and Törnqvist- price index index were consistently considerably lower than the COLI based on a demand system for the same data. We also saw that many tests for the assumptions of the respective models failed and many results were such that they did not really make sense. It may be suggested that these unsatisfactory findings may perhaps be attributed to the specific German data. Even if this were the case it is still true, that estimating a demand system of more than only some few goods ($N = 9$ in our case) entails so many insurmountable difficulties that it would by no means be a reasonable option for official statistics to compile a CPI based on estimated demand systems. Thus a monthly COLI-type official CPI based on estimated demand systems comprising a tolerable variety of goods will most probably be impossible.

Even worse, there may be indications that the assumptions underlying the often praised (alleged) microeconomic foundation of the COLI are unrealistic. This can be inferred from the unsatisfactory fit of our (demand) regression equations. In the first place we may conjecture that in reality households can hardly respond so promptly and rapidly to price signals by substituting as assumed in theory. Such theory related arguments should be kept in mind when we consider the superlative-index approach next, because for this approach to be valid it has to rely on the same assumptions (concerning consumer behaviour) and to some more in addition (concerning the notion of "approximating" an "arbitrary" function).

As already mentioned in this situation it appears tempting to avoid all those econometric difficulties with demand systems by simply calculating a superlative index combining observable data vectors $\mathbf{p}_t, \mathbf{p}_0, \mathbf{q}_0$ and $\mathbf{q}_t$ only (where perhaps only the timely availability of $\mathbf{q}_t$ may pose a problem in practice). This, however, is not that easy. The proof of "superlative-ness" of an index function requires the restrictive assumptions discussed in section 2 which are unlikely to hold empirically, and together with the assumptions needed to relate the index to a utility maximum this makes the index no less dependent on restrictive and perhaps unrealistic assumptions than the much simpler COGI approach.

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A Appendix

Table 1: AIDS estimates for 1988

coeff.	estimator	standard errors	t-stat	p-value	sig.[1)]
α_1	0.3478	0.0531	6.5600	0.0000	***
α_2	-0.0072	0.0296	-0.2400	0.8080	
α_3	0.2819	0.0336	8.3800	0.0000	***
α_4	0.0996	0.0469	2.1200	0.0340	*
α_5	-0.0085	0.0185	-0.4600	0.6460	
α_6	0.0388	0.0276	1.4100	0.1600	
α_7	0.0292	0.0249	1.1700	0.24102	
α_8	0.1645	0.0294	5.6000	0.0000	***
α_9	0.0539	0.0627	0.8600	0.3900	
β_1	0.0495	0.0084	5.8700	0.0000	***
β_2	-0.0076	0.0045	-1.6800	0.0930	.
β_3	-0.0036	0.0046	-0.7700	0.4390	
β_4	-0.0019	0.0062	-0.3000	0.7640	
β_5	-0.0159	0.0027	-5.8900	0.0000	***
β_6	0.0000	0.0059	0.0000	0.9990	
β_7	-0.0124	0.0039	-3.1500	0.0020	**
β_8	-0.0093	0.0073	-1.2700	0.2030	
β_9	0.0012	0.0090	0.1300	0.8970	
γ_{11}	0.1509	0.0227	6.6400	0.0000	***
γ_{12}	-0.0198	0.0097	-2.0500	0.0410	*
γ_{13}	-0.0243	0.0095	-2.5600	0.0110	*
γ_{14}	0.0192	0.0152	1.2700	0.2060	
γ_{15}	-0.0185	0.0059	-3.1600	0.0020	**
γ_{16}	-0.0005	0.0087	-0.0600	0.9540	
γ_{17}	-0.0185	0.0080	-2.3100	0.0210	
γ_{18}	-0.0192	0.0086	-2.2300	0.0260	***
γ_{19}	-0.0693	0.0172	-4.0200	0.0000	***
γ_{22}	0.0130	0.0081	1.6200	0.1060	
γ_{23}	-0.0067	0.0056	-1.2000	0.2310	
γ_{24}	-0.0124	0.0089	-1.3900	0.1650	
γ_{25}	-0.0014	0.0035	-0.4000	0.6860	
γ_{26}	0.0052	0.0050	1.0500	0.2920	

γ_{27}	-0.0059	0.0047	-1.2400	0.2150	
γ_{28}	0.0001	0.0047	0.0200	0.9850	
γ_{29}	0.0278	0.0096	2.8900	0.0040	**
γ_{33}	0.0803	0.0077	10.3700	0.0000	***
γ_{34}	-0.0090	0.0086	-1.0500	0.2960	
γ_{35}	-0.0073	0.0035	-2.0900	0.0360	*
γ_{36}	-0.0167	0.0049	-3.3800	0.0010	***
γ_{37}	-0.0026	0.0047	-0.5600	0.5750	
γ_{38}	-0.0107	0.0047	-2.2500	0.0250	*
γ_{39}	-0.0032	0.0095	-0.3300	0.7390	
γ_{44}	-0.0018	0.0202	-0.0900	0.9300	
γ_{45}	0.0110	0.0056	1.9500	0.0510	.
γ_{46}	0.0002	0.0071	0.0200	0.9820	
γ_{47}	-0.0040	0.0073	-0.5500	0.5810	
γ_{48}	-0.0080	0.0066	-1.2100	0.2250	
γ_{49}	0.0049	0.0146	0.3300	0.7390	
γ_{55}	0.0261	0.0031	8.4000	0.0000	***
γ_{56}	-0.0055	0.0030	-1.8500	0.0640	.
γ_{57}	-0.0028	0.0029	-0.9700	0.3300	
γ_{58}	-0.0031	0.0028	-1.0900	0.2760	
γ_{59}	0.0016	0.0058	0.2700	0.7830	
γ_{66}	0.0154	0.0072	2.1400	0.0320	*
γ_{67}	-0.0010	0.0042	-0.2300	0.8180	
γ_{68}	-0.0013	0.0053	-0.2400	0.8070	
γ_{69}	0.0042	0.0090	0.4600	0.6440	
γ_{77}	0.0330	0.0056	5.9300	0.0000	***
γ_{78}	-0.0080	0.0040	-1.9700	0.0490	*
γ_{79}	0.0097	0.0080	1.2200	0.2210	
γ_{88}	0.0543	0.0082	6.6500	0.0000	***
γ_{89}	-0.0042	0.0089	-0.4700	0.6380	
γ_{99}	0.0285	0.0230	1.2400	0.2140	

number of iterations: 3

[1] level of significance: . = 0.1, *=0.05, **=0.01, ***=0.001.

Table 2: QUAIDS estimates for 1988

coeff.	estimator	standard errors	t-stat	p-value	sig.[1)]
α_1	0.3491	0.0523	6.6700	0.0000	***
α_2	-0.0081	0.0296	-0.2700	0.7860	
α_3	0.2811	0.0342	8.2100	0.0000	***
α_4	0.0987	0.0465	2.1200	0.0340	*
α_5	-0.0095	0.0186	-0.5100	0.6120	
α_6	0.0391	0.0276	1.4200	0.1570	
α_7	0.0298	0.0248	1.2000	0.2300	
α_8	0.1679	0.0291	5.7800	0.0000	***
α_9	0.0518	0.0622	0.8300	0.4050	*
β_1	0.0610	0.0123	4.9600	0.0000	***
β_2	-0.0075	0.0057	-1.3000	0.1940	
β_3	-0.0109	0.0070	-1.5500	0.1210	
β_4	-0.0090	0.0087	-1.0300	0.3010	
β_5	-0.0209	0.0043	-4.8400	0.0000	***
β_6	0.0009	0.0074	0.1200	0.9080	
β_7	-0.0089	0.0053	-1.6800	0.0920	.
β_8	-0.0012	0.0102	-0.1200	0.9060	
β_9	-0.0035	0.0116	-0.3000	0.7610	
γ_{11}	0.1470	0.0227	6.4800	0.0000	***
γ_{12}	-0.0209	0.0097	-2.1600	0.0310	*
γ_{13}	-0.0225	0.0095	-2.3600	0.0180	*
γ_{14}	0.0217	0.0152	1.4300	0.1530	
γ_{15}	-0.0178	0.0059	-3.0300	0.0020	**
γ_{16}	-0.0005	0.0087	-0.0600	0.9510	
γ_{17}	-0.0193	0.0080	-2.4100	0.0160	*
γ_{18}	-0.0185	0.0086	-2.1500	0.0310	*
γ_{19}	-0.0692	0.0172	-4.0200	0.0000	***
γ_{22}	0.0129	0.0081	1.5900	0.1110	
γ_{23}	-0.0063	0.0056	-1.1200	0.2640	
γ_{24}	-0.0115	0.0089	-1.2900	0.1980	
γ_{25}	-0.0011	0.0035	-0.3000	0.7630	
γ_{26}	0.0052	0.0050	1.0500	0.2930	
γ_{27}	-0.0062	0.0047	-1.3000	0.1920	
γ_{28}	-0.0001	0.0047	-0.0100	0.9920	
γ_{29}	0.0278	0.0096	2.8800	0.0040	**

γ_{33}	0.0796	0.0077	10.2800	0.0000	***
γ_{34}	-0.0100	0.0086	-1.1600	0.2440	
γ_{35}	-0.0075	0.0035	-2.1800	0.0290	*
γ_{36}	-0.0166	0.0049	-3.3800	0.0010	***
γ_{37}	-0.0022	0.0047	-0.4800	0.6310	
γ_{38}	-0.0110	0.0047	-2.3300	0.0200	*
γ_{39}	-0.0034	0.0095	-0.3500	0.7230	
γ_{44}	-0.0039	0.0202	-0.1900	0.8460	
γ_{45}	0.0106	0.0056	1.8800	0.0610	.
γ_{46}	0.0001	0.0071	0.0200	0.9860	
γ_{47}	-0.0035	0.0073	-0.4800	0.6290	
γ_{48}	-0.0086	0.0066	-1.2900	0.1970	
γ_{49}	0.0051	0.0146	0.3500	0.7250	
γ_{55}	0.0261	0.0031	8.4200	0.0000	***
γ_{56}	-0.0055	0.0030	-1.8500	0.0640	.
γ_{57}	-0.0028	0.0029	-0.9400	0.3460	
γ_{58}	-0.0035	0.0028	-1.2300	0.2200	
γ_{59}	0.0015	0.0058	0.2600	0.7980	
γ_{66}	0.0154	0.0072	2.1500	0.0320	*
γ_{67}	-0.0010	0.0042	-0.2300	0.8180	
γ_{68}	-0.0013	0.0053	-0.2400	0.8090	
γ_{69}	0.0041	0.0090	0.4600	0.6450	
γ_{77}	0.0329	0.0056	5.9100	0.0000	***
γ_{78}	-0.0078	0.0040	-1.9200	0.0550	.
γ_{79}	0.0098	0.0080	1.2300	0.2170	
γ_{88}	0.0551	0.0082	6.7400	0.0000	***
γ_{89}	-0.0044	0.0089	-0.5000	0.6190	
γ_{99}	0.0286	0.0230	1.2500	0.2120	
λ_1	-0.0277	0.0155	-1.7900	0.0740	.
λ_2	-0.0004	0.0085	-0.0500	0.9580	
λ_3	0.0176	0.0088	2.0000	0.0450	*
λ_4	0.0174	0.0114	1.5200	0.1280	
λ_5	0.0119	0.0051	2.3500	0.0190	*
λ_6	-0.0021	0.0109	-0.1900	0.8490	
λ_7	-0.0084	0.0073	-1.1400	0.2540	
λ_8	-0.0195	0.0135	-1.4500	0.1470	
λ_9	0.0113	0.0165	0.6800	0.4950	

number of iterations: 3
[1] level of significance: . = 0.1, *=0.05, **=0.01, ***=0.001.

Table 3: coefficients of determination of the AIDS and QUAIDS estimation for 1988

equation	AIDS	QUAIDS
w_1	0.0839	0.0869
w_2	0.0134	0.0131
w_3	0.1002	0.1037
w_4	0.0051	0.0073
w_5	0.0991	0.1040
w_6	0.0038	0.0039
w_7	0.0464	0.0474
w_8	0.0428	0.0444

Table 4: LR test of the linear restrictions of the AIDS

model 1: symmetry and homogeneity restrictions
model 2: only homogeneity restrictions
model 3: no restrictions

Modell		df[1]	log lik.[2]	ddf[3]	LR	p-value	sig.[4]
1988	1	52	12,641.22				
	2	80	12,663.46	28	44.48	0.0249	**
1988	1	52	12,641.22				
	3	88	12,667.90	36	53.36	0.0312	**
1993	1	52	7,708.46				
	2	80	7,724.41	28	31.91	0.2782	
1993	1	52	7,708.46				
	3	88	7,733.36	36	49.79	0.0629	*
1998	1	52	4,738.93				
	2	80	4,758.69	28	39.52	0.0729	*
1998	1	52	4,738.93				
	3	88	4,767.06	36	56.25	0.0170	**
2003	1	52	3,475.45				
	2	80	3,496.34	28	41.79	0.0454	**
2003	1	52	3,475.45				
	3	88	3,504.09	36	57.28	0.0135	**

[1] degrees of freedom
[2] log likelihood value
[3] difference between degrees of freedom of the two models
[4] level of significance: * = 0.1, **=0.05, ***=0.01

Table 5: LR test of the linear restrictions of the QUAIDS

model 1: symmetry and homogeneity restrictions
model 2: only homogeneity restrictions
model 3: no restrictions

Modell		df[1)]	log lik.[2)]	ddf[3)]	LR	p-value	sig.[4)]
1988	1	60	12,649.86				
	2	88	12,672.38	28	45.04	0.0219	**
1988	1	60	12,649.86				
	3	96	12,676.81	36	53.90	0.0280	**
1993	1	60	7,711.36				
	2	88	7,727.61	28	32.50	0.2548	
1993	1	60	7,711.36				
	3	96	7,736.91	36	51.10	0.0491	**
1998	1	60	4,754.24				
	2	88	4,775.68	28	42.88	0.0358	**
1998	1	60	4,754.24				
	3	96	4,783.80	36	59.12	0.0089	***
2003	1	60	3,494.32				
	2	88	3,516.31	28	43.98	0.0279	**
2003	1	60	3,494.32				
	3	96	3,523.93	36	59.22	0.0087	***

1) degrees of freedom
2) log likelihood value
3) difference between degrees of freedom of the two models
4) level of significance: * = 0.1, **=0.05, ***=0.01

Table 6: percentage of observations not violating the monotonicity and concavity conditions

data set	model	monotonicity	concavity	
			fitted	oberserved
1988	AIDS	100%	0%	0%
	QUAIDS	100%	0%	0%
1993	AIDS	100%	0.158%	0%
	QUAIDS	100%	1.738%	0%
1998	AIDS	100%	7.635%	0.246%
	QUAIDS	100%	5.665%	0.248%
2003	AIDS	100%	5.068%	1.351%
	QUAIDS	100%	2.703%	3.378%

Table 7: LR of the AIDS vs. the QUAIDS

model 1: AIDS with symmetry and homogeneity restrictions
model 2: QUAIDS with symmetry and homogeneity restrictions

Modell		df[1)]	log lik.[2)]	ddf[3)]	LR	p-value	sig.[4)]
1988	1	52	12,641.22				
	2	60	12,649.86	8	17.28	0.0273	**
1993	1	52	7,708.46				
	2	60	7,711.36	8	5.80	0.6696	
1998	1	52	4,738.93				
	2	60	4,754.24	8	30.62	0.0001	***
2003	1	52	3,475.45				
	2	60	3,494.32	8	37.74	0.0000	***

1) degrees of freedom
2) log likelihood value
3) difference between degrees of freedom of the two models
4) level of significance: * = 0.1, **=0.05, ***=0.01

Table 8: COLIs and Laspeyres-price index, base year 1988

Jahr	AIDS	QUAIDS	Laspeyres
1988	1.0000	1,0000	1.0000
1989	1.0037	1.0062	1.0185
1990	1.0075	0.9978	1.0396
1991	1.0109	0.9989	1.0615
1992	1.0131	1.0047	1.0710
1993	1.0099	0.9968	1.0526
1994	1.0200	1.0168	1.1064
1995	1.0249	1.0347	1.1319
1996	1.0247	1.0272	1.1320
1997	1.0287	1.0423	1.1536
1998	1.0371	1.0706	1.2024
1999	1.0293	1.0478	1.1568
2000	1.0369	1.0601	1.2017
2001	1.0436	1.0638	1.2437
2002	1.0428	1.0555	1.2401
2003	1.0413	1.0515	1.2323
2004	1.0396	1.0427	1.2236
2005	1.0436	1.0559	1.2475
2006	1.0466	1.0635	1.2662
2007	1.0576	1.0870	1.3322
2008	1.0688	1.1028	1.4142
2009	1.0552	1.0828	1.3269

Table 9: COLIs and superlative price indices, base year 1988

year	Laspeyres	Paasche	Fisher	Törnqvist	AIDS	QUAIDS
1993	1.0526	1.0566	1.0546	1.0547	1.0099	0.9968
1998	1.2024	1.2052	1.2038	1.2038	1.0371	1.0706
2003	1.2323	1.2562	1.2442	1.2439	1.0413	1.0515